JOURNAL OF FUNCTIONAL ANALYSIS 71, 1–46 (1987)

Twisted Perron–Frobenius Theorem and L-Functions

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Communicated by the Editors

Received December 18, 1985

A theorem of Perron–Frobenius type and its twisted version are established in a setting of topological graphs. The applications include a partial extension of Selberg's results on his zeta functions and a result by Parry and Pollicott on meromorphic continuations of dynamical zeta functions to certain L-functions associated to a dynamical system of Anosov type. © 1987 Academic Press, Inc.

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INTRODUCTION

The classical notion of L-function in number theory has several interesting analogues in geometry. The most celebrated one is the zeta function introduced by A. Selberg [21], which is a function in one complex variable associated with a representation of a discrete subgroup of $PSL_2(\mathbb{R})$. To be more exact, for a finite dimensional unitary representation ρ of a co-compact torsion free discrete subgroup Γ, the Selberg zeta function is defined as the product

$$Z(s, \rho) = \prod_{k=0}^{\infty} \prod_{\mathfrak{p}} \det(I - \rho(\langle \mathfrak{p} \rangle) \exp(-(s+k)\, l(\mathfrak{p}))),$$

where $\mathfrak{p}$ runs over all prime closed geodesics in the Riemann surface $SO(2)\backslash PSL_2(\mathbb{R})/\Gamma$, $l(\mathfrak{p})$ denotes the length of $\mathfrak{p}$, and $\langle \mathfrak{p} \rangle$ denotes an element in Γ whose conjugacy class corresponds to the free homotopy class of the closed curve $\mathfrak{p}$. Selberg proved, among other things, that $Z(s, \rho)$ converges absolutely in $\operatorname{Re} s > 1$, and has an analytic continuation to the whole plane. The most striking fact is that the function $Z(s, \rho)$ nearly satisfies the

* Supported by the Ishida Foundation.

0022-1236/87 $3.00

"Riemann hypothesis," and the zeros of $Z(s, \rho)$ are completely described in terms of the eigenvalues of the Laplacian acting on the sections of the flat vector bundle associated with the representation ρ.

Instead of $Z(s, \rho)$, we consider the ratio

$$L(s, \rho) = Z(s+1, \rho)/Z(s, \rho) = \prod_{\mathfrak{p}} \det(I - \rho(\langle \mathfrak{p} \rangle) N(\mathfrak{p})^{-s})^{-1}, \tag{1}$$

where $N(\mathfrak{p}) = \exp l(\mathfrak{p})$. The function $L(s, \rho)$ is a more direct analogue of the Artin L-function of Galois extensions of number fields, and is more convenient when we apply a similar argument as in number theory to the study of closed geodesics. In fact, we may prove, in much the same way as in analytic number theory, analogues of the prime ideal theorem and the Chebotarev density theorem (see [8, 20, 25]).

Taking account of the fact that prime closed geodesics are the images of closed orbits of the geodesic flow on the unit tangent bundle of the Riemann surface, we are naturally led to a notion of L-function in general dynamical systems. Let (X, ϕ_t) be a flow on a compact manifold X. We define $L(s, \rho)$ by the right hand side of (1), where, in this turn, $\mathfrak{p}$ runs over all closed orbits, $l(\mathfrak{p})$ denotes the least period of $\mathfrak{p}$ and ρ is a unitary representation of the fundamental group of X. It should be noted that the dynamical zeta function introduced by D. Ruelle [19] is just $L(s, \mathbb{1})$, where $\mathbb{1}$ is the trivial representation (see also S. Smale [23]).

A primary purpose of this paper is to show the following partial generalization of Selberg's results mentioned above.

THEOREM A. *Let (X, ϕ_t) be a smooth Anosov flow on a compact smooth manifold X, and let $h > 0$ be the topological entropy of (X, ϕ_t). If there exists a smooth invariant measure on X or ϕ_t has X as its nonwandering set, then*

(1) *$L(s, \rho)$ converges absolutely and is holomorphic when* $\operatorname{Re} s > h$.

(2) *$L(s, \rho)$ has a nowhere vanishing meromorphic extension to an open neighborhood of* $\operatorname{Re} s \geq h$.

(3) *$L(s, \mathbb{1})$ has a simple pole at $s = h$.*

(4) *For a character χ, $L(s, \chi)$ has a pole at $s = h + \sqrt{-1}\, t$ if and only if $\chi(\mathfrak{p}) = \exp \sqrt{-1}\, tl(\mathfrak{p})$ for all $\mathfrak{p}$. In this case, $L(s, \chi) = L(s - \sqrt{-1}\, t, \mathbb{1})$, and every pole on* $\operatorname{Re} s = h$ *is simple.*

We should point out that if $\rho = \mathbb{1}$ (the case of zeta function), or more generally if the image of ρ is of finite order, then Theorem A reduces to the theorems established by Parry and Pollicott [13, 14]. For general ρ, we must accomplish elaborate analysis of "twisted" Ruelle operators defined on certain topological graphs, which, in a sense, play a role similar to the

Laplacians acting on sections of flat vector bundles. In some special cases, one can prove analyticity of $L(s, \rho)$ on $\operatorname{Re} s = h$.

THEOREM B. *Let φ: $\pi_1(X) \to G$ be a homomorphism such that the image of those elements whose conjugacy classes contain closed orbits generates G. Then for any irreducible representation ρ: $G \to U(N)$ with $N \geq 2$, $L(s, \rho \circ \varphi)$ is holomorphic in some neighborhood of* $\operatorname{Re} s \geq h$.

A case which satisfies the condition in Theorem B is the geodesic flow on the unit tangent bundle UM of a compact Riemannian manifold M, where $G = \pi_1(M)$ and φ: $\pi_1(UM) \to G$ is the homomorphism induced by the projection: $UM \to M$. In fact, this comes from the well known fact that each conjugacy class in $\pi_1(M)$ contains at least one closed geodesic. Furthermore one can show that if ρ: $\pi_1(X) \to U(N)$, $N \geq 2$, is irreducible and the image is of finite order, then $L(s, \rho)$ is holomorphic (see the remark in Section 6). In case of characters, Theorem A(4) assures that if a non-trivial character χ has the image of finite order and if (X, ϕ_t) is topologically mixing, then $L(s, \chi)$ is holomorphic in a neighborhood of $\operatorname{Re} s \geq h$. These observations lead to the following theorem of Chebotarev type by using a routine method in analytic number theory (cf. Parry and Pollicott [14]).

PROPOSITION C. *Let (X, ϕ_t) be a topologically mixing Anosov flow, and let Γ be a normal subgroup in $\pi_1(X)$ of finite index. Then for any conjugacy class $[g]$ in the quotient group $\pi_1(M)/\Gamma$, we have*

$$\lim_{x \to \infty} \frac{hx}{e^{hx}} \# \{\mathfrak{p};\, l(\mathfrak{p}) < x,\, \pi(\mathfrak{p}) \in [g]\} = \frac{\# [g]}{\# (\pi_1(X)/\Gamma)},$$

where π: $\pi_1(X) \to \pi_1(X)/\Gamma$ denotes the projection.

We now introduce an analogue of L-function in graph theory, which can be regarded as an analogue of Selberg zeta function for one-dimensional spaces and turns out to be useful in proving Theorem A. Let (V, E) be an *oriented graph* with V, the set of *vertices*, and E, the set of *edges*, so that $E \subset V \times V$. It is assumed that a positive function $l(e)$ is assigned to each edge e, which we call the *length* of e. For each edge $e = (u, v)$, we set $o(e) = u$, $t(e) = v$. An element of the form $c = (e_1, ..., e_n)$ such that $e_i \in E$ and $t(e_i) = o(e_{i+1})$, $i = 1, ..., n-1$ is called a *path* in (V, E). We then put $l(c) = l(e_1) + \cdots + l(e_n)$. A path $c = (e_1, ..., e_n)$ is called *closed* if $t(e_n) = o(e_1)$, and called *prime* if, in addition, there is no divisor k of n such that $1 \leq k < n$ and $e_{i+k} = e_i$ for $i \in \mathbb{Z}/n\mathbb{Z}$. Two closed paths $c = (e_1, ..., e_n)$ and $c' = (e'_1, ..., e'_n)$ are equivalent if there is an integer k such that $e'_i = e_{i+k}$ for $i \in \mathbb{Z}/n\mathbb{Z}$. A *prime cycle* is, by definition, an equivalence class of a prime closed path, which is denoted by $\mathfrak{p}$. The length of $\mathfrak{p}$ is defined in an obvious way, and denoted $l(\mathfrak{p})$.

A graph which we shall mainly treat is a *projective limit* of finite graphs: $(V, E) = \varprojlim (V_n, E_n)$ $(n = 1, 2, ...)$. The *fundamental group* of (V, E) is defined as the projective limit $\pi_1(V, E) = \varprojlim \pi_1(V_n, E_n)$, where $\pi_1(V_n, E_n)$ is the fundamental group of the one-dimensional CW-complex associated with (V_n, E_n). Note that each prime cycle $\mathfrak{p}$ defines, in a natural way, a conjugacy class in $\pi_1(V, E)$, from which we select a representative $\langle \mathfrak{p} \rangle$. Associated with a continuous unitary representation ρ: $\pi_1(V, E) \to U(N)$, the L-function $L(s, \rho)$ is defined in the same manner as (1), where the product is taken over prime cycles in (V, E).

An aspect on which we lay stress is that the projective limits V and E have natural topology. In fact, we may supply metrices on V and E defined by

$$d(u, v) = \theta^{\sup\{n;\, \hat{\omega}_n(u) = \hat{\omega}_n(v)\}}$$

$$d(e, e') = \max\{d(o(e), o(e')), d(t(e), t(e'))\},$$

where $0 < \theta < 1$, and $\hat{\omega}_n$: $(V, E) \to (V_n, E_n)$ denotes the projection. We then have the following.

THEOREM D. *Suppose that* (i) *each finite graph* (V_n, E_n) *is irreducible*; (ii) *the morphisms* ω_n: $(V_n, E_n) \to (V_{n-1}, E_{n-1})$ *are surjective, and* ω_n: $o^{-1}(v) \to o^{-1}(\omega_n(v))$ *is bijective for any* $v \in V_n$; *and* (iii) *if* $\omega_n(e) = \omega_n(e')$, *then* $l(e) = l(e')$. *We also suppose that the length function l is Lipschitz continuous with respect to the metric d. If (V, E) is not a circuit graph, then there exists a positive constant h such that $L(s, \rho)$ satisfies the same properties* (1)–(4) *as in Theorem A. Furthermore, if ρ is irreducible and* $\dim \rho \geqslant 2$, *then $L(s, \rho)$ is holomorphic in a neighborhood of* $\operatorname{Re} s \geqslant h$.

A typical example of projective limit of finite graph satisfying the conditions in Theorem D is a one-sided shift of finite type; that is, the space $\Sigma^+(V, E)$ of infinite step paths $c = (e_1, e_2, ...)$ in an irreducible finite graph (V, E). Edges in $\Sigma^+(V, E)$ are those pairs $(c, c') \in \Sigma^+(V, E) \times \Sigma^+(V, E)$ with $\sigma(c') = c$, where σ: $\Sigma^+(V, E) \to \Sigma^+(V, E)$ denotes the shift operator: $\sigma(e_1, e_2, ...) = (e_2, e_3, ...)$. In fact, if we define finite graphs (V_n, E_n) by

$$V_n = \{c = (e_1, ..., e_n);\ \text{paths in } (V, E)\}$$

$$E_n = \{(c, c') \in V_n \times V_n;\ e_2' = e_1, ..., e_n' = e_{n-1}\}$$

then we easily find that $\Sigma^+(V, E) \simeq \varprojlim (V_n, E_n)$ (see Example 3 in Section 2). We shall call (V_n, E_n) the graph of n-step paths.

We now give a brief explanation about how Theorem D is related to Theorem A. Given an Anosov flow (X, ϕ_t), we construct a Markov family of sufficiently small size, and associate a subshift $\Sigma^+(V, E)$ of finite type and a positive function f on $\Sigma^+(V, E)$ which is cohomologous to the suspending function of Bowen's symbolic dynamical system. If we set $l(e) = f(\ell(e))$, then the function l is Lipschitz continuous in the above sense for some $\theta \in (0, 1)$. Since the graph (V, E) can be embedded in X as a CW-complex, one has a homomorphism $\pi_1(V, E) \to \pi_1(X)$, which, by composition

$$\pi_1(\Sigma^+(V, E)) \longrightarrow \pi_1(V_1, E_1) \longrightarrow \pi_1(V, E) \longrightarrow \pi_1(X) \xrightarrow{\rho} U(N),$$

induces a representation $\bar{\rho}$ of $\pi_1(\Sigma^+(V, E))$. Here $\pi_1(V_1, E_1) \to (V, E)$ is a homomorphism induced by the orientation-reversing morphism $(V_1, E_1) \to (V, E)$ given by $e \to o(e)$. Let $L(s, \bar{\rho})$ be the L-function associated with the profinite graph $\Sigma^+(V, E)$, the length function l and the representation $\bar{\rho}$. Applying a similar idea due to R. Bowen [2], we find

$$L(s, \rho) = L(s, \bar{\rho})\, g(s),$$

where $g(s)$ is a product of L-functions and reciprocals of L-functions of the auxiliary subshifts. The constant h in Theorem C for the L-function $L(s, \bar{\rho})$ coincides with the topological entropy of (X, ϕ_t). Since $g(s)$ turns out to be a non-vanishing holomorphic function in a neighborhood of $\operatorname{Re} s \geqslant h$, we obtain Theorem A.

The detailed contents of the paper are as follows. In Section 1, we give an idea of the proof of Theorem D by treating the case of finite graphs. In Section 2, we introduce the notion of topological graphs, which gives us a natural setting for generalization of the Perron–Frobenius–Ruelle theorem as well as the twisted version. Profinite graphs satisfying the conditions in Theorem D constitute a special class of topological graphs. Section 3 starts with the definition of covering maps of topological graphs. Since topological graphs are, in general, highly disconnected (for instance, the graph corresponding to a one-sided shift is totally disconnected), one cannot introduce the notion of universal coverings. But we still have a "nice" formulation of covering maps which fits in with graph structure and topological structure (this is one of reason why we lay stress on graph structure of one-sided subshifts). We then pass to generalized Ruelle operators acting on sections of a "flat" vector bundles associated with a unitary representation of a covering transformation group. A theorem of Perron–Frobenius type which concerns the case of the trivial representation is established in Section 3. In Section 4, we investigate the spectrum of Ruelle operators in case of non-trivial representations. Section 5 is devoted to the proof of Theorem D, in which we confine ourselves to pro-finite graphs. The fundamental idea is due to Ruelle, Parry and Pollicott, who mainly treat analytic continuations of zeta functions corresponding to the trivial representation. In Section 6, we apply the results in Section 5 to the L-function associated to Anosov flows and prove Theorem A.

Recently much attention has been given to extending the meromorphic domains of zeta functions including the dynamical zeta functions. See Pollicott [17] and Fried [4].

We should point out that Y. Ihara [10] introduced a zeta function associated to a p-adic curve, which can be interpreted in terms of a finite graph.

1. L-Functions of Finite Graphs

In this section, we illustrate Theorem B by giving a proof for the case of finite graphs. Some of the results and ideas will be employed later on in general settings.

Let (V, E) be an irreducible finite graph with a length function l. Irreducibility means that every two vertices of V can be joined by a path in (V, E). We always assume that (V, E) is not a circuit graph. Here a circuit graph is a graph isomorphic to $(\mathbb{Z}/n\mathbb{Z}, \{(i, i+1);\ i \in \mathbb{Z}/n\mathbb{Z}\})$. Given a path $c = (e_1, ..., e_n)$, we write $|c|$ for n. For a prime cycle $\mathfrak{p}$, we define $|\mathfrak{p}|$ likewise. The *origin* (resp. *terminus*) of c is $o(e_1)$ (resp. $t(e_n)$), which we denote by $o(c)$ (resp. $t(c)$). We also write $c(j) = o(e_{j+1})$ $(j = 0, 1, ..., n-1)$, and $c(n) = t(e_n)$. The composition $c \cdot c'$ is defined in an obvious way for paths c and c' with $t(c) = o(c')$.

Let $|(V, E)|$ denote the CW-complex associated with the graph (V, E), which is arcwise connected in view of irreducibility of (V, E). A graph associated with the universal covering of $|(V, E)|$ is denoted by $(\tilde{V}, \tilde{E})$, where the orientation of edges is determined in such a way that the map ω: $\tilde{V} \to V$ induced from the covering map is a morphism of graphs. It is known that $(\tilde{V}, \tilde{E})$ is a *tree*, so that there is at most one path joining two vertices in $(\tilde{V}, \tilde{E})$. (See J. P. Serre [22] for basic properties of trees.) The fundamental group $\pi_1(V, E)$ of the CW-complex $|(V, E)|$ acts on $(\tilde{V}, \tilde{E})$ in a natural manner as orientation-preserving automorphisms of the graph. Lifting the function l, we have a length function on $(\tilde{V}, \tilde{E})$, which we also denote l.

We define the operator L_s: $\mathrm{Map}(\tilde{V}, \mathbb{C}^N) \to \mathrm{Map}(\tilde{V}, \mathbb{C}^N)$ by setting

$$(L_s \varphi)(x) = \sum_{\substack{e \in \tilde{E} \\ o(e) = x}} \exp(-sl(e))\, \varphi(t(e)).$$

Given a representation ρ: $\pi_1(V, E) \to U(N)$, we define the finite dimensional subspace S_ρ by

$$S_\rho = \{\varphi \in \mathrm{Map}(\tilde{V}, \mathbb{C}^N);\ \varphi(\gamma x) = \rho(\gamma)\,\varphi(x) \text{ for } \gamma \in \pi_1(V, E) \text{ and } x \in \tilde{V}\},$$

which, as is easily seen, is left invariant under L_s, and $\dim S_\rho = \# V \times N$. For simplicity, we write $L_{s,\rho}$ for $L_s | S_\rho$. The matrix $L_{s,\rho}$ is obviously holomorphic with respect to $s \in \mathbb{C}$. When $\rho = \mathbb{1}$, the trivial representation, the operator $L_{s,\mathbb{1}}$ is identified with an endomorphism of $\mathrm{Map}(V, \mathbb{C})$. From the Perron–Frobenius theorem, if $s \in \mathbb{R}$, there exists a maximum positive simple eigenvalue $\lambda(s)$ for the operator $L_{s,\mathbb{1}}$ with an associated positive eigenfunction $u_s \in \mathrm{Map}(V, \mathbb{R})$. The following characterization of $\lambda(s)$ is useful:

$$\lambda(s) = \max\{\lambda;\ \text{for some non-negative } \varphi \neq 0,\ L_{s,\mathbb{1}}\,\varphi \geqslant \lambda\varphi\}.$$

Since $\lambda(s)$ is a strictly decreasing function of s and $\lim_{s \to \infty} \lambda(s) = 0$, $\lim_{s \to -\infty} \lambda(s) = \infty$, we may find a unique positive h such that $\lambda(h) = 1$ (positivity of h is a consequence of the hypothesis that (V, E) is non-circuit). We first show

LEMMA 1-1. (1) *If* $\mathrm{Re}\, s > h$, *then* 1 *is not an eigenvalue of* $L_{s,\rho}$.

(2) *If* $N = \dim \rho \geqslant 2$ *and* ρ *is irreducible, then* 1 *is not an eigenvalue of* $L_{s,\rho}$ *for any* s *with* $\mathrm{Re}\, s = h$.

(3) *Let* $s = h + \sqrt{-1}\, t$, *and let* χ *be a character. Then* 1 *is an eigenvalue of* $L_{s,\chi}$ *if and only if* $\chi(\langle \mathfrak{p} \rangle) = \exp \sqrt{-1}\; t l(\mathfrak{p})$ *for all prime cycle* $\mathfrak{p}$ *in* (V, E).

Proof. Suppose that $L_{s,\rho}\varphi = \lambda\varphi$ for some non-zero $\varphi \in S_\rho$. Taking the norm of the both sides, we have the inequality

$$L_{\mathrm{Re}\,s,\mathbb{1}} \|\varphi\| \geqslant |\lambda|\, \|\varphi\|,$$

where we should note that $\|\varphi\| \in S_{\mathbb{1}}$ since $\|\varphi(\gamma x)\| = \|\rho(\gamma)\,\varphi(x)\| = \|\varphi(x)\|$. Hence $|\lambda| \leqslant \lambda(\mathrm{Re}\, s)$, proving the first assertion. If $s = h + \sqrt{-1}\, t$ and $\lambda = 1$, then $L_{h,\mathbb{1}}\|\varphi\| = \|\varphi\|$ and $\|\varphi\| = C u_h$ for some positive constant C. Applying the convexity argument (cf. [12]), we get

$$(\exp -\sqrt{-1}\; t l(e))\ \varphi(t(e))/u_h(t(e)) = \varphi(o(e))/u_h(o(e)).$$

This implies that there exists a function $a(x, y) \in \mathbb{C}$ such that $|a(x, y)| = 1$ and

$$\varphi(y)/u_h(y) = a(x, y)\ \varphi(x)/u_h(x).$$

It is easily checked that $\{a(x, y)\}$ satisfies

$$\begin{aligned} a(x, x) &= 1 \\ a(x, y)\, a(y, z) &= a(x, z) \qquad\qquad (1\text{-}1) \\ a(o(c), t(c)) &= \exp \sqrt{-1}\, tl(c). \end{aligned}$$

In particular, we have

$$\begin{aligned} \rho(\gamma)\, \varphi(x) = \varphi(\gamma x) &= a(x, \gamma x)\, u_h(\gamma x)\, u_h(x)^{-1} \varphi(x) \\ &= a(x, \gamma x)\, \varphi(x), \end{aligned}$$

from which it follows that the one-dimensional subspace $\mathbb{C}\varphi(x)$ in $\mathbb{C}^N$ is invariant under the ρ-action. This implies (2), and also that if $\rho = \chi$, then $\chi(\langle \mathfrak{p} \rangle) = \exp \sqrt{-1}\, tl(\mathfrak{p})$ for all $\mathfrak{p}$.

To show the "if" part of (3), we fix a base point x_0 in $\tilde{V}$, and define a function φ on $\tilde{V}$ by setting

$$\varphi(x) = a(x_0, x)\, u_h(x)/u_h(x_0),$$

where $a(x, y)$ is a function uniquely determined by the relation (1-1). We then have

$$\begin{aligned} (L\, \varphi)(x) &= \sum_{t(e) = x} \exp(-(h + \sqrt{-1}\, t)\, l(e))\, a(x_0, x) \exp \sqrt{-1}\, tl(e) \\ &\qquad \times u_h(t(e))/u_h(x_0) \\ &= \sum_{t(e) = x} a(x_0, x) \exp(-hl(e))\, u_h(t(e))/u_h(x_0) \\ &= a(x_0, x)\, u_h(x)/u_h(x_0) = \varphi(x), \end{aligned}$$

where we have used the equality $a(x_0, t(e)) = a(x_0, x)\ a(x, t(e)) = \exp \sqrt{-1}\, tl(e)\, a(x_0, x)$. It now remains to prove that $\varphi(\gamma x) = \chi(\gamma)\, \varphi(x)$ for all γ. We first assume γ has the form $\langle \mathfrak{p} \rangle$. This being the case, there exist paths c_1 and c_2 such that $t(c_2) = \langle \mathfrak{p} \rangle\, o(c_2)$, $o(c_1) = x$, and $t(c_1) = o(c_2)$. We then observe

$$\begin{aligned} \varphi(\gamma x) &= a(x_0, x)\, a(x, \gamma x)\, u_h(\gamma x)/u_h(x_0) \\ &= a(x_0, x) \exp \sqrt{-1}\, tl(c_1) \exp \sqrt{-1}\, tl(\gamma c_1) \\ &\qquad \times \exp \sqrt{-1}\, tl(c_2)\, u_h(x)/u_h(x_0) \\ &= \exp \sqrt{-1}\, tl(\mathfrak{p})\, a(x_0, x)\, u_h(x)/u_h(x_0) \\ &= \chi(\gamma)\, \varphi(x). \end{aligned}$$

For a general γ, it suffices to note that there exist $\gamma_1, \ldots, \gamma_k \in \pi_1(V, E)$ such that each γ_i is conjugate to $\langle \mathfrak{p}_i \rangle$ for some prime cycle $\mathfrak{p}_i$, and $\gamma = \gamma_1^{\pm 1} \cdots \gamma_k^{\pm 1}$ (see [1] for proof).

LEMMA 1-2. *$L(s, \rho) = \det(I - L_{s,\rho})^{-1}$ for* $\operatorname{Re} s > h$. *In particular, $L(s, \rho)$ has a meromorphic continuation to the whole plane.*

Proof. We first compute the trace of $(L_{s,\rho})^k$. For this, we take a basis of S_ρ in the following way. Let $\{e_n\}_{n=1}^N$ be the standard basis of $\mathbb{C}^N$, and $\mathscr{D}$ be a fundamental subset in V for the π_1-action, namely,

$$\tilde{V} = \bigcup_{\gamma \in \pi_1} \gamma \mathscr{D} \qquad \text{and} \qquad \gamma \mathscr{D} \cap \mathscr{D} = \phi \qquad \text{for} \quad \gamma \neq \mathrm{id}.$$

We then set, for $x \in \mathscr{D}$ and $n = 1, \ldots, N$,

$$\varphi_{xn}(y) = \begin{cases} \rho(\gamma)\, e_n & \text{if} \quad y = \gamma x \\ 0 & \text{otherwise,} \end{cases}$$

which turn out to be a basis of S_ρ. Computing the matrix elements of $L_{s,\rho}^k$ with respect to this basis, we find

$$\operatorname{tr}(L_{s,\rho}^k) = \sum_{\gamma \in \pi_1} \sum_{x \in \mathscr{D}} \operatorname{tr} \rho(\gamma)\, L_K(s; x, \gamma x), \tag{1-2}$$

where

$$L_K(s; x, y) = \begin{cases} \exp -sl(c) & \text{if there is a path } c \text{ in } (\tilde{V}, \tilde{E}) \\ & \text{with } |c| = k,\ o(c) = x \text{ and } t(c) = y \\ 0 & \text{otherwise} \end{cases}$$

Now suppose $L_K(s; x, \gamma x) \neq 0$. The image $\omega(\tilde{c})$ of the path $\tilde{c}$ joining x and γx is a closed path in (V, E). Conversely for a closed path c in (V, E) one can find a unique pair $(x, \gamma) \in \mathscr{D} \times \pi_1$ with $L_k(s; x, \gamma x) \neq 0$. Hence, the right hand side of (1-2) equals

$$\sum_{c: |c| = k} \operatorname{tr} \rho(\langle c \rangle) \exp -sl(c), \tag{1-3}$$

where $\langle c \rangle \in \pi_1(V, E)$ is a representative of the conjugacy class corresponding to the free homotopy class of the closed path c. Since a closed path can be expressed in a unique way as an iteration of a prime closed path, it follows that (1-3) equals

$$\sum_{\substack{m,\mathfrak{p} \\ m|\mathfrak{p}|=k}} |\mathfrak{p}| \operatorname{tr} \rho(\langle \mathfrak{p} \rangle^m) \exp - sm\, l(\mathfrak{p}),$$

so

$$\log L(s,\rho) = \sum_{m=1}^{\infty} \sum_{\mathfrak{p}} m^{-1} \operatorname{tr} \rho(\langle \mathfrak{p} \rangle^m) \exp - sm\, l(\mathfrak{p})$$

$$= \sum_{k=1}^{\infty} k^{-1} \operatorname{tr}(L_{s,\rho}^k) = \log \det(I - L_{s,\rho})^{-1}.$$

This completes the proof.

The other assertions about the location of poles are easy consequences of Lemma 1-1.

Remark. It is interesting to note that the computation of $\operatorname{tr}(L_{s,\rho})^k$ bears resemblance to that of the trace of invariant integral operators on a locally symmetric space by means of Selberg trace formula, in which case closed geodesics play a role similar to closed paths (see [21, 27]).

2. Topological Graphs

To give a natural setting for generalizing L-functions of finite graphs, we introduce a class of graphs with topology. We shall start with providing some basic conditions, which, as we shall see in the next sections, allow all the usual results on eigenvalues of non-negative matrices to be transferred to "infinite dimensional matrices" associated with such graphs.

Let (V, d) be a complete metric space and $E \subset V \times V$ a closed set. We define a distance function on $V \times V$ by

$$d((u,v),(u',v')) = \max\{d(u,u'), d(v,v')\}.$$

If the following three conditions hold, then the triple (V, E, d) is called a *topological graph:*

(TG1) The maps $o\colon E \to V$ $(e \to o(e))$ and $t\colon E \to V$ $(e \to t(e))$ are surjective.

(TG2) There exists a positive constant δ such that for $e \in E$, $o(B_\delta(e)) = B_\delta(o(e))$ and the restriction $o \mid B_\delta(e)\colon B_\delta(e) \to B_\delta(o(e))$ is an isometry, where $B_\delta(e)$ denotes the closed ball of radius δ with center e. In particular $o\colon E \to V$ is a covering map of topological spaces.

(TG3) There exists a constant $A \in (0, 1)$ such that if $d(e, e') \leqslant \delta$, then $d(t(e), t(e')) \leqslant A d(e, e')$ (or equivalently $d(t(e), t(e)) \leqslant A d(o(e), o(e')))$.

A topological graph (V, E, d) is said to be compact if V is compact, and said to be *irreducible if for every open set* U_1 and U_2 in V, there exists a path c with $o(c) \in U_1$ and $t(c) \in U_2$. It should be noted that if a topological graph (V, E, d) is compact, then $\sup_{v \in V} \#(o^{-1}(v))$ is finite.

We now exhibit several examples of topological graphs.

EXAMPLE 1. A graph (V, E) with the discrete distance function is a topological graph.

EXAMPLE 2. Given a sequence of graphs (V_n, E_n), $n \geqslant 1$, and morphisms of graphs ω_n: $(V_n, E_n) \to (V_{n-1}, E_{n-1})$, we construct a new graph (V, E) as follows:

$$V = \varprojlim V_n = \left\{ v = (v_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} V_n ; \omega_n(v_n) = v_{n-1} \right\}$$

$$E = \varprojlim E_n = \{ (u, v) \in V \times V ; (u_n, v_n) \in E_n \}.$$

We denote by $\tilde{\omega}_n$: $(V, E) \to (V_n, E_n)$ the canonical projection. We supply a distance function d_θ $(0 < \theta < 1)$ defined by

$$d_\theta(u, v) = \theta^{\sup\{n; u_n = v_n\}}.$$

Note that the distance on E coincides with the distance given by

$$d_\theta(e, e') = \theta^{\sup\{n; e_n = e'_n\}}.$$

It is easily observed that (V, E, d_θ) is a topological graph if the following conditions are satisfied:

(PLG1) ω_n: $V_n \to V_{n-1}$ is surjective for $n \geqslant 2$.

(PLG2) ω_n: $o^{-1}(v) \to o^{-1}(\omega_n(v))$ is bijective for $v \in V_n$ and $n \geqslant 2$.

(PLG3) If $\omega_n(e) = \omega_n(e')$, then $t(e) = t(e')$.

(PLG4) For each $n \geqslant 1$, o: $E_n \to V_n$ and t: $E_n \to V_n$ are surjective.

In fact, (PLG2) and (PLG4) guarantee the condition (TG1). The condition (TG2) is a consequence of (PLG2) and (PLG3), where we should put $\delta = \theta$. (TG3) comes from (PLG3) with $A = \theta$. We shall call $(\varprojlim V_n, \varprojlim E_n)$ satisfying (PLG1)–(PLG4) a projective limit graph. If, in addition, each (V_n, E_n) is a finite graph, then $(\varprojlim V_n, \varprojlim E_n)$ is called a *pro-finite graph*, which is obviously compact. The conditions (PLG3) and (PLG4) guarantee that if (V_1, E_1) is finite, then every (V_n, E_n) is finite. In fact this is inductively proved in the following way. Suppose that (V_{n-1}, E_{n-1}) is finite. For each $u \in \omega_n^{-1}(v)$, $v \in V_{n-1}$, select an edge e_u with $\ell(e_u) = u$. Then $\omega_n(e_u) \in \ell^{-1}(v)$, and by (PLG3), $\omega_n(e_u) \neq \omega_n(e_{u'})$, whenever $u \neq u'$. Thus we have the inequality

$$\begin{aligned}\# V_n = \sum_{v \in V_{n-1}} \# \omega_n^{-1}(v) \leqslant \sum_{v \in V_{n-1}} \# \ell^{-1}(v) = \sum_{v \in V_{n-1}} \# o^{-1}(v)\\ \leqslant (\sup_{v \in V_{n-1}} \# o^{-1}(v))(\# V_{n-1}),\end{aligned}$$

hence (V_n, E_n) is finite. Since (PLG2) implies that

$$\sup_{v \in V_{n-1}} \# o^{-1}(v) = \sup_{v \in V_1} \# o^{-1}(v),$$

we have the following useful inequality.

$$\# V_n \leqslant A^{n-1}(\# V_1), \qquad A = \sup_{v \in V_1} \# o^{-1}(v). \tag{2-1}$$

Furthermore, if (V_1, E_1) is irreducible, then every (V_n, E_n) as well as $(\varprojlim V_n, \varprojlim E_n)$ are irreducible.

It follows from (PLG 2) and (PLG3) that the correspondence $c \to \tilde{\omega}_n(c)$ gives a bijection of the set of closed paths in (V, E) onto the set of closed paths in (V_n, E_n).

Conversely suppose that the distance function d on a topological graph (V, E, d) satisfies the "non-archimedean" property

$$d(u, v) \leqslant \max\{d(u, w), d(v, w)\} \qquad (u, v, w \in V).$$

(Note that the distance function d_θ introduced above satisfies this property.) This being the case, the relations $\underset{n}{\sim}$ on V and E defined by

$$\begin{aligned} u \underset{n}{\sim} v &\leftrightarrow d(u, v) \leqslant \delta^n\\ e \underset{n}{\sim} e' &\leftrightarrow d(e, e') \leqslant \delta^n \end{aligned}$$

are equivalence relations (we assume here that $\delta < 1$). Let V_n and E_n be the sets of equivalence classes. We then easily find that (V_n, E_n), $n \geqslant 1$, constitutes a sequence of graphs with canonical morphisms $(V_n, E_n) \to (V_{n-1}, E_{n-1})$, and that the projections $V \to V_n$ and $E \to E_n$ give rise to an isomorphism $(V, E) = \varprojlim(V_n, E_n)$. We also observe that the distance d_δ on (V, E) induced by this isomorphism satisfies the inequality

$$\delta\, d_\delta(u, v) < d(u, v) \leqslant d_\delta(u, v)$$

for $u, v \in V$ with $d(u, v) \leqslant \delta$. Furthermore if $A \leqslant \delta$, then $\{(V_n, E_n)\}$ satisfies (PLG1)-(PLG4).

EXAMPLE 3. This is a sub-example of Example 2. Let (V, E) be a finite irreducible graph, and (V_n, E_n), $n \geqslant 1$, be the graph of n-step paths in (V, E) (see the Introduction). We set $\omega_n(e_1,..., e_n) = (e_1,..., e_{n-1})$. Then $o^{-1}(e_1,..., e_n) = \{(e_1,..., e_n) \times (e, e_1,..., e_{n-1}) \in V_n \times V_n;\quad e \in E\}$ so that (PLG2) is obviously satisfied. Suppose that $\omega_n((e_1,..., e_n) \times (f_1,..., f_n)) = \omega_n((e'_1,..., e'_n) \times (f'_1,..., f'_n))$ in E_{n-1}. Then $f_1 = f'_1,..., f_{n-1} = f'_{n-1}$ and $f_n = e_{n-1} = e'_{n-1} = f'_n$; hence (PLG3) follows. The conditions (PLG1) and (PLG4) are immediate. Thus $\{(V_n, E_n)\}$ yields a pro-finite graph $(\varprojlim V_n, \varprojlim E_n, d_\theta)$. Clearly the elements of $\varprojlim V_n$ are identified with elements in the one-sided shift $\Sigma^+(V, E)$ and $(u, v) \in \varprojlim V_n \times \varprojlim V_n$ is an edge if and only if $\sigma(v) = u$, where σ denotes the shift operator of $\Sigma^+(V, E)$.

One can construct many examples of pro-finite graphs which are not isomorphic to graphs of one-sided shifts.

We now return to the general case. Let (V, E, d) be a topological graph, and let c be a path in (V, E). By virtue of the conditions (TG2) and (TG3), one can find, for each point x in V with $d(x, o(c)) \leqslant \delta$, a unique path c' such that $o(c') = x$, $|c| = |c'|$ and $d(c(j), c'(j)) \leqslant \Lambda^j d(o(c), o(c'))$ for $1 \leqslant j \leqslant |c|$. We will call c' a *companion* of c with the origin x.

LEMMA 2-1. *Let (V, E, d) be a topological graph and let c be a path in (V, E) such that $|c| = n$ and $d(t(c), o(c)) \leqslant \delta$. Then there exists a closed path c_∞ such that $|c_\infty| = n$ and*

$$d(c_\infty(j), c(j)) \leqslant \Lambda^j (1 - \Lambda^n)^{-1} d(t(c), o(c)), \qquad j = 0, 1,..., n.$$

Proof. Take a sequence $\{c_k\}_{k=0}^\infty$ of paths in such a way that $c_0 = c$ and c_{k+1} is a companion of c_k with $o(c_{k+1}) = t(c_k)$. We then have

$$d(c_{k+1}(j), c_k(j)) \leqslant \Lambda^{nk+j} d(t(c), o(c)), \qquad j = 0, 1,..., n,$$

so that $\{c_k(j)\}_{k=0}^\infty$ is a Cauchy sequence in V. If we set $c_\infty(j) = \lim_{k \to \infty} c_k(j)$, then $c_\infty(n) = c_\infty(0)$. Since E is closed in $V \times V$, we find that c_∞ is eventually a closed path, and

$$\begin{aligned} d(c_\infty(j), c(j)) &\leqslant \sum_{k=0}^\infty d(c_k(j), c_{k+1}(j)) \\ &\leqslant \Lambda^j (1 - \Lambda^n)^{-1} d(t(c), o(c)). \end{aligned}$$

This completes the proof.

LEMMA 2-2. *If (V, E, d) is a compact topological graph, then there exists a positive constant C such that the number of closed paths c in (V, E) with $|c| = n$ is less than C^n for any $n \geqslant 1$.*

Proof Choose points $x_1, \ldots, x_a \in V$ so that V is covered by open balls $U_{\delta/2}(x_i)$, $i = 1, \ldots, a$. Define a map

$$\Phi\colon \{c; \text{closed paths with } |c| = n\} \to \prod_{k=1}^{n} \{1, 2, \ldots, a\}$$

by putting $\Phi(c) = (j_1, \ldots, j_n)$, where j_i is the smallest j with $c(i) \in U_{\delta/2}(x_j)$. Suppose $\Phi(c) = \Phi(c')$. Then

$$d(c(i), c'(i)) \leqslant d(c(i), x_{j_i}) + d(x_{j_i}, c'(i)) < \delta.$$

In view of the condition (TG3), we have

$$d(c(i), c'(i)) \leqslant A^n\, d(c(i), c'(i)),$$

so that $c(i) = c'(i)$ for all i. This implies that Φ is injective, whence the lemma.

We denote by $\Omega_x(n)$ the set of paths c with $|c| = n$ and $o(c) = x$.

LEMMA 2-3. *Irreducibility of (V, E, d) is equivalent to the condition that, for any x in V, the set $t(\bigcup_{n=1}^{\infty} \Omega_x(n))$ is dense in V.*

Proof It suffices to prove that irreducibility implies that for every x, y in V and positive ε there exists a path c with $o(c) = x$ and $d(t(c), y) < \varepsilon$. Let c' be a path such that

$$d(o(c'), x) < \min(\varepsilon/2, \delta)$$

$$d(t(c'), y) < \varepsilon/2$$

and let c be a companion of c' with $o(c) = x$. Then we have

$$\begin{aligned} d(t(c), y) &\leqslant d(y, t(c')) + d(t(c'), t(c)) \\ &< \frac{\varepsilon}{2} + A^{|c|} \frac{\varepsilon}{2} \leqslant \varepsilon, \end{aligned}$$

as required.

LEMMA 2-4. *If (V, E, d) is compact and irreducible, then there exists an infinite path $c = (e_1, e_2, \ldots)$ such that $\{e_i\}$ is dense in E.*

Proof From compactness it follows that $\Omega_x(n)$ is a finite set; hence in view of Lemma 2–3 the spaces V and E are separable. We let $\{U_m\}_{m=1}^{\infty}$ be an open base of E. Choose e_1' in U_1. From irreducibility, we may find a path c_1 such that $o(c_1) = t(e_1')$ and $t(c_1) \in o(U_2)$. We then take $e_2' \in U_2$ and a path c_2 such that $o(e_2') = t(c_1)$ and $o(c_2) = t(e_2')$. Continuing this process, we get an infinite path $c=e_1'\cdot c_1\cdot e_2'\cdot c_2\cdot\cdots$ which possess the desired property.

In what follows, we assume that (V, E, d) is compact and irreducible. We set

$$\Omega_{x,\eta}=\{c;\text{ paths with } o(c)=x \text{ and } d(t(c), x)\leqslant \eta\}.$$

We also denote by $\nu(x,\eta)$ the greatest common divisor of the set of integers $\{|c|;\, c\in\Omega_{x,\eta}\}$.

LEMMA 2-5. *The number $\nu(x,\eta)$ does not depend on the choice of x and η, where $0<\eta\leqslant\delta$.*

Proof. We first show that $\nu(x,\eta)=\nu(y,\eta)$. Let c and c' be paths satisfying

$$o(c)=x,\qquad d(t(c), y)<\eta/2,$$

$$o(c')=y\qquad d(t(c'), x)<\eta/2 \text{ and } |c'|>-2/\log \Lambda.$$

If c'' be the companion of c' with $o(c'')=t(c)$, then

$$\begin{aligned} d(t(c\cdot c''), x)&\leqslant d(x, t(c'))+d(t(c'), t(c''))\\ &<\frac{\eta}{2}+d(o(c'), o(c''))<\eta, \end{aligned}$$

so $c\cdot c''\in\Omega_{x,\eta}$ and $\nu(x,\eta)$ divides $|c|+|c''|$. Let $c_0\in\Omega_{y,\eta}$, and let c_0' be the companion of c_0 with $o(c_0')=t(c)$ and c''' be the companion of c' with $o(c''')=t(c_0')$. Then

$$\begin{aligned} d(t(c\cdot c_0'\cdot c'''), x)&\leqslant d(x, t(c'))+d(t(c'), t(c'''))\\ &<\frac{\eta}{2}+\frac{\eta}{2}\Lambda^{|c'|}(1+\Lambda^{|c_0|})<\eta, \end{aligned}$$

hence $c\cdot c_0'\cdot c'''\in\Omega_{x,\eta}$. Consequently $\nu(x,\eta)\mid(|c|+|c_0'|+|c'''|)$ and $\nu(x,\eta)\mid |c_0|$ since $\nu(x,\eta)\mid(|c|+|c'''|)$. This implies $\nu(x,\eta)\mid\nu(y,\eta)$. Changing the roles of x and y, we have $\nu(x,\eta)=\nu(y,\eta)$.

We next check that $\nu(x,\eta)=\nu(x,\eta')$ for $0<\eta'<\eta\leqslant\delta$. It is obvious that $\nu(x,\eta)\mid\nu(x,\eta')$. Let $c\in\Omega_{x,\eta}$ and let $c'\in\Omega_{x,\eta'/2}$ with $\Lambda^{|c'|}\eta<\eta'/2$. If c'' is the companion of c' with $o(c'')=t(c)$, then

$$\begin{aligned} d(t(c\cdot c''), x)&\leqslant d(t(c''), t(c'))+d(t(c'), x)\\ &<\Lambda^{|c'|}\eta+\frac{\eta'}{2}<\eta', \end{aligned}$$

so $c \cdot c'' \in \Omega_{x,\eta'}$. Since $\nu(x,\eta') \mid |c''|$, we have $\nu(x,\eta') \mid |c|$; hence $\nu(x,\eta') \mid \nu(x,\eta)$.

We call the number $\nu(x,\eta)$ the period of (V, E, d), which we denote hereafter by $\nu(V, E, d)$. If $\nu(V, E, d) = 1$, the graph (V, E, d) is called primitive. When (V, E) is a finite graph, this is the same as saying that the zero-one matrix $A = (a_{x,y})$, $(a_{x,y} = 1 \Leftrightarrow (x, y) \in E)$, is aperiodic in the sense that $A^n > 0$ for some $n \geqslant 1$. In case $\nu(V, E, d) > 1$, we may decompose V into disjoint subsets in the following way. We set

$$V(x,j) = (y \in V \mid \text{there exists a path } c \in \textstyle\bigcup_{n=1}^{\infty} \Omega_x(n\nu + j)$$
$$\text{with } d(t(c), y) \leqslant \delta),$$
$$j = 0, 1, ..., \nu - 1 = \nu(V, E, d) - 1.$$

We then have

(i) each $V(x,j)$ is open and closed,

(ii) $V = V(x,0) \cup V(x,1) \cup \cdots \cup V(x, \nu - 1)$ (disjoint).

Indeed, from the definition it follows immediately that each $V(x,j)$ is closed, and that V is the union of $V(x,j)$, $j = 0, 1, ..., \nu - 1$. Suppose that $V(x,i) \cap V(x,j) \neq \varnothing$. Then one can find paths c_1 and c_2 such that

$$|c_1| \equiv i \quad (\text{mod } \nu), \qquad d(t(c_1), y) \leqslant \delta,$$
$$|c_2| \equiv j \quad (\text{mod } \nu), \qquad d(t(c_2), y) \leqslant \delta,$$
$$o(c_1) = o(c_2) = x.$$

We let c_3 be a path such that $o(c_3) = y$ and $d(t(c_3), x) < \delta/2$. Suppose that $|c_3|$ is large enough. If c_1' and c_2' are the companions of c_3 with $o(c_1') = t(c_1)$ and $o(c_2') = t(c_2)$, respectively, then

$$d(t(c_1 \cdot c_1'), x) \leqslant d(x, t(c_3)) + d(t(c_3), t(c_2'))$$
$$\leqslant \frac{\delta}{2} + A^{|c_3|}\delta < \delta$$

and likewise $d(t(c_2 \cdot c_2'), x) < \delta$, so

$$|c_1| + |c_3| \equiv 0 \qquad (\text{mod } \nu)$$
$$|c_2| + |c_3| \equiv 0 \qquad (\text{mod } \nu).$$

Thus we have that

$$i \equiv |c_1| \equiv |c_2| \equiv j \qquad (\text{mod } \nu),$$

which implies (ii) and also that $V(x,j)$ is open.

By similar argument, we easily check that each $V(x, j)$ is an equivalence class of the following equivalence relation:

$x \sim y$ if and only if there exists a path c with $v \mid |c|$, $o(c) = x$ and $d(t(c), y) \leqslant \delta$.

We shall call $V(x, j)$ a primitive part of V.

LEMMA 2-6. *Let $V = V_1 \cup V_2 \cup \cdots \cup V_v$ be the decomposition into the primitive parts. If points x and y lie in the same V_i, then for any $\varepsilon > 0$, there exists a natural number n_0 such that for any $n \geqslant n_0$*

$$t(\Omega_x(nv)) \cap B_\varepsilon(y) \neq \varnothing.$$

in particular, for any sequence $\{x_n\}_{n=1}^{\infty} \subset V_i$, the union $t(\bigcup_{n=1}^{\infty} \Omega_{x_n}(nv))$ is dense in V_i.

Proof. We may suppose $\varepsilon < \delta$. From irreducibility and what we have said above, it follows that there exists a path $c \in \Omega_x(mv)$ with $d(t(c), y) < \varepsilon/2$ for some m. We choose paths $c_1, c_2, \ldots, c_t \in \Omega_{y, \varepsilon(1-A)/2}$ such that $|c_k| = s_k v$ and the greatest common divisor of $\{s_1, \ldots, s_t\}$ is one. From Lemma 2-1, there exist closed paths $c_{k\infty}$ such that $|c_{k\infty}| = s_k v$ and

$$d(c_k(j), c_{k\infty}(j)) \leqslant A^j (1 - A^{s_k v})^{-1} \varepsilon (1 - A)/2 < \varepsilon/2.$$

In view of the assumption on $\{s_1, \ldots, s_t\}$ we have made, we may find a natural number n_1 such that for $n \geqslant n_1$ there exist positive integers $q_1, \ldots, q_t$ satisfying

$$\sum_{k=1}^{t} q_k s_k = n \qquad \text{and} \qquad tA^{q_k} < \tfrac{1}{2}.$$

We now inductively define paths $c'_1, c'_2, \ldots, c'_t$ in such a way that c'_1 is the companion of $(c_{1\infty})^{q_1} = c_{1\infty} \cdots c_{1\infty}$ (q_1-times iteration of $c_{1\infty}$) with $o(c'_1) = t(c)$ and c'_k is the companion of $(c_{k\infty})^{q_k}$ with $o(c'_k) = t(c'_{k-1})$. Then

$$|c \cdot c'_1 \cdots c'_t| = \left(m + \sum_{k=1}^{t} q_k s_k\right) v,$$

and

$$d(t(c \cdot c'_1 \cdots c'_t), y) < \left(\sum_{k=1}^{t} A^{q_k}\right) \varepsilon + \frac{\varepsilon}{2} < \varepsilon.$$

Hence setting $n_0 = m + n_1$, we have the result.

Remark. For a pro-finite graph ($\varprojlim V_n$, $\varprojlim E_n$, d_θ), the primitive decomposition is given by

$$\varprojlim V_n = \bigcup_{j=1}^{\nu} \{v = (v_n);\, v_1 \in V_{1j}\},$$

where $V_1 = \bigcup_{j=1}^{\nu} V_{1j}$ is the primitive decomposition of (V_1, E_1).

3. Ruelle Operators for Topological Graphs

This section and the next are concerned with a generalization of Lemma -1. We first introduce a notion of covering maps for topological graphs, and then define Ruelle operators acting on sections of a "flat" vector bundle associated with a unitary representation of the covering transformation group. Some of results in the case of the trivial representation are proved along the same line as in Pollicott [16].

Let $(\hat{V}, \hat{E}, \hat{d})$ and (V, E, d) be topological graphs. A morphism π: $(\hat{V}, \hat{E}) \to (V, E)$ is called a *covering* map if the following conditions are satisfied.

(C-1) π: $\hat{V} \to V$ is surjective, and for all $\hat{x} \in \hat{V}$, the restrictions of π

$$\pi: o^{-1}(\hat{x}) \to o^{-1}(\pi(\hat{x}))$$
$$\pi: t^{-1}(\hat{x}) \to t^{-1}(\pi(\hat{x}))$$

are bijective. This, in particular, implies that the induced map π: $|(\hat{V}, \hat{E})| \to |(V, E)|$ is a covering map of CW-complexes.

(C-2) There exists a positive ε such that for every $\hat{x} \in \hat{V}$, the restriction π: $B_\varepsilon(\hat{x}) \to B_\varepsilon(\pi(\hat{x}))$ is an isometry and surjective. In particular, π: $\hat{V} \to V$ is a covering map of topological space in the usual sense.

It should be noted that the constants δ and A in the conditions (TG2) and (TG3) are chosen in common for both $(\hat{V}, \hat{E})$ and (V, E). An example of covering map will be given at the end of the next section.

We let G be the group of isometries g of $(\hat{V}, \hat{E})$ such that $\pi \circ g = \pi$. We call G the covering transformation group of the covering. If G acts transitively on each fiber of π, the covering map is called normal. From the condition (C-1), it follows that for each path c in (V, E) and a point $\hat{x}$ in $\hat{V}$ with $\pi(\hat{x}) = o(c)$, there exists a unique path $\hat{c}$ in $(\hat{V}, \hat{E})$ such that $\pi(\hat{c}) = c$ and $o(\hat{c}) = \hat{x}$, which we call a lift of c. If π is normal and c is closed, there exists a unique $\langle c \rangle \in G$ such that $t(\hat{c}) = \langle c \rangle\, o(\hat{c})$. It is easily checked that $\langle c \rangle$ is uniquely determined by c up to conjugacy.

We adopt the following notations. For a $\mathbb{C}^N$-valued function f on a metric space (X, d), we set

$$\|f\|_\infty = \sup_{x \in X} \|f(x)\| \equiv \sup_{x \in X} \{|f_1(x)|^2 + \cdots + |f_N(x)|^2\}^{1/2}$$

$$\mathrm{Lip}(f) = \sup_{x \in X} \|f(x) - f(y)\|/d(x, y)$$

$$\|f\|_1 = \|f\|_\infty + \mathrm{Lip}(f).$$

We denote by $C^1(X, \mathbb{C}^N)$ the Banach space of continuous functions f with $\|f\|_1 < \infty$. It is easily shown that $\|f \cdot g\|_1 \leqslant \|f\|_1 \, \|g\|_1$.

Let π: $(\hat{V}, \hat{E}, \hat{d}) \to (V, E, d)$ be a normal covering map with covering transformation group G, and let ρ: $G \to U(N)$ be a unitary representation. The quotient space $\hat{V} \times \mathbb{C}^N/G$ by a natural action of G on the product space $\hat{V} \times \mathbb{C}^N$ has a structure of vector bundle on V, which we denote by F_ρ. A section s of F_ρ is identified with a function $\hat{s}$: $\hat{V} \to \mathbb{C}^N$ satisfying the relation $\hat{s}(\gamma x) = \rho(\gamma)\, \hat{s}(x)$. We denote by $C^1(F_\rho)$ the space of section s of F_ρ such that $\hat{s} \in C^1(\hat{V}, \mathbb{C}^N)$. The norm of $s \in C^1(F_\rho)$ is defined by

$$\|s\|_1 = \|\hat{s}\|_1 = \|\hat{s}\|_\infty + \mathrm{Lip}(\hat{s}).$$

It should be noted that when $\rho = \mathbb{1}$, the trivial representation, $C^1(F_\rho)$ is identified with $C^1(V, \mathbb{C})$.

From now on we assume that (V, E, d) is compact and irreducible. We set $M = \sup_{v \in V} \# o^{-1}(v)$. Let $f \in C^1(E, \mathbb{C})$. We define the operator L_f: $C^1(\hat{V}, \mathbb{C}^N) \to C^1(\hat{V}, \mathbb{C}^N)$ by

$$L_f\, \hat{s}(x) = \sum_{e: o(e) = x} f(\pi(e))\, \hat{s}(t(e)).$$

If we set $f(c) = f(e_1) \cdots f(e_n)$ for a path $c = (e_1, \ldots, e_n)$, then we find

$$(L_f)^n \hat{s}(x) = \sum_{\substack{c: |c| = n \\ o(c) = x}} f(\pi(c))\, \hat{s}(t(c)).$$

The following is easy to check.

LEMMA 3-1. *$L_f(C^1(F_\rho)) \subset C^1(F_\rho)$, and the restriction L_f: $C^1(F_\rho) \to C^1(F_\rho)$ is a continuous operator.*

We set $L_{f,\rho} = L_f \mid C^1(F_\rho)$, and call the twisted Ruelle operator. The following is straightforward.

LEMMA 3-2. *The linear mapping $f \mapsto L_{f,\rho}$ is continuous with respect to* *the* $\|\cdot\|_1$*-norm, namely there exists a constant* $C>0$ *such that* $\|L_{f,\rho}s\|_1 \leqslant C\|f\|_1 \|s\|_1$ *for any* $f\in C^1(E)$ *and* $s\in C^1(F_\rho)$.

Our main concern is the spectrum of $L_{f,\rho}$. An important feature of the Ruelle operators is that the spectrum does not change essentially by the following scaling transformation: Let $u\in C^1(V, \mathbb{C})$ be nowhere vanishing and let λ be a non-zero complex number. We set

$$\bar{f}(e)=\lambda^{-1}f(e)\,u(t(e))\,u(o(e))^{-1},$$

which we call the scaling transformation of f by u and λ. We see immediately that the diagram

$$\begin{array}{ccc} C^1(F_\rho) & \xrightarrow{\lambda^{-1}L_{f,\rho}} & C^1(F_\rho) \\ {\scriptstyle \times u^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \times u^{-1}} \\ C^1(F_\rho) & \xrightarrow[L_{\bar{f},\rho}]{} & C^1(F_\rho), \end{array}$$

is commutative so that the spectrum of $L_{\bar{f},\rho}$ is $\lambda^{-1}\times$(the spectrum of $L_{f,\rho}$)

In this section, we mainly treat the case $\rho=\mathbb{1}$. We set $L_f=L_{f,\mathbb{1}}$ for simplicity. Under the identification $C^1(F_1)=C^1(V,\mathbb{C})$, we have

$$L_f g(x)=\sum_{e;\,o(e)=x} f(e)\,g(t(e)).$$

Remark. If (V, E) is a one-sided subshift $\Sigma^+(V, E)$ and $f(e)=F(t(e))$ for some function F on $\Sigma^+(V, E)$, then

$$L_f g(x)=\sum_{y;\,\sigma y=x} F(y)\,g(y).$$

The right hand side is a usual definition of the Ruelle operator acting on functions defined on the one-sided subshift. (See D. Ruelle [18].)

LEMMA 3-3. *If* $f\in C^1(E)$ *is positive valued, then the operator* L_f *has a positive simple eigenvalue with a positive eigenfunction.*

This lemma can be shown in much the same way as in [16, 18], but for the reader's convenience, we give a proof.

We set

$$r=(\min f)^{-1}\,\mathrm{Lip}(f)$$

$$C=r(1-A)^{-1},$$

and let K denote the $\|\cdot\|_\infty$-closed convex set of non-negative continuous functions g on V with $\|g\|_\infty \leqslant 1$ and $g(x) \leqslant g(y)\exp(Cd(x, y))$ whenever $d(x, y) \leqslant \delta$. For every $g \in K$, we have

$$|g(x) - g(y)| \leqslant e^{Cd(x, y)} - 1$$

for x, y with $d(x, y) \leqslant \delta$; hence K is a convex subset of $C^1(V)$ and $\|\cdot\|_\infty$-compact by the Ascoli–Arzela theorem. Define a continuous map I_n: $K \to C^1(V)$ by

$$I_n(g) = L_f\left(g + \frac{1}{n}\right) \Big/ \left\| L_f\left(g + \frac{1}{n}\right)\right\|_\infty, \qquad n = 1, 2, \ldots.$$

If $d(x, y) \leqslant \delta$, we have the inequalities

$$\begin{aligned} f(e)/f(e') &= 1 + (f(e) - f(e'))/f(e') \\ &\leqslant 1 + rd(x, y) \leqslant e^{rd(x, y)} \\ g(\ell(e)) &\leqslant g(\ell(e')) \exp(Cd(\ell(e), \ell(e')) \\ &\leqslant g(\ell(e')) \exp(CAd(x, y)), \end{aligned}$$

where $o(e) = x$ and e' is the companion of e with $o(e') = y$. Thus we have

$$\begin{aligned} L_f\left(g + \frac{1}{n}\right)(x) &= \sum_{e;\, o(e) = x} f(e)\left(g(\ell(e)) + \frac{1}{n}\right) \\ &\leqslant \sum_{e';\, o(e') = y} \exp(r\, d(x, y))\, f(e')\left(g(\ell(e')) + \frac{1}{n}\right) \\ &\quad \times \exp(CA\, d(x, y)) \\ &\leqslant L_f\left(g + \frac{1}{n}\right)(y) \exp(C\, d(x, y)) \end{aligned}$$

and $I_n g \in K$. By the Schauder–Tychonoff theorem, I_n has a fixed point u_n in K. We put

$$\lambda_n = \left\| L_f\left(u_n + \frac{1}{n}\right)\right\|_\infty$$

and choose a $\|\cdot\|_\infty$-convergent subsequence $\{u_{n_j}\}$. If we set $u = \lim u_{n_j}$ and $\lambda = \lim \lambda_{n_j}$, then $L_f u = \lambda u$ and

$$\lambda_{n_j} u_{n_j} \geqslant \min(f)\left\{\min(u_{n_j}) + \frac{1}{n_j}\right\}.$$

so $\lambda \geqslant \min(f) > 0$. The eigenfunction u is strictly positive, for if $u(x_0) = 0$ for some point x_0 in V, then

$$0 = \lambda^n u(x_0) = L_f^n u(x_0) = \sum_c f(c)\, u(\ell(c)),$$

where c runs over the paths with $|c| = n$ and $o(c) = x_0$. Hence $u = 0$ on the dense subset $\ell(\bigcup_{n=1}^{\infty} \Omega_{x_0}(n))$, which yields a contradiction since $\|u\|_\infty = 1$.

We finally check that λ is simple. Suppose that a real valued function $k \in C^1(V)$ satisfies $L_f k = \lambda k$. Set $b = \min(k/u)$. Then $(k - bu)(x_0) = 0$ for some x_0. Repeating the same argument as above, we get $k = bu$. This completes the proof.

Let u and λ be the eigenfunction and the eigenvalue obtained above, and $\bar{f}$ be the scaling transformation of f by u and λ. Then $L_{\bar{f}} 1 = 1$. This simple observation is useful in the following discussions.

LEMMA 3-4. *Let $f \in C^1(E)$ be nowhere vanishing, and suppose that $L_{|f|} 1 =$. Then there exists a positive constant C such that for any $s \in C^1(F_\rho)$*

$$\mathrm{Lip}(L_{f,\rho}^n s) \leqslant C \|s\|_\infty + \Lambda^n \,\mathrm{Lip}(s), \qquad n \geqslant 1.$$

Proof. From the assumption, it follows that $\|f\|_\infty \leqslant 1$. Since

$$\|L_f^n \hat{s}(\hat{x})\| \leqslant \sum_{\substack{c;\, o(c) = \hat{x} \\ |c| = n}} |f(\pi(c))|\, \|\hat{s}(\ell(c))\| \leqslant \|\hat{s}\|_\infty,$$

we have, in the case $\hat{d}(\hat{x}, \hat{y}) \geqslant \delta$,

$$\|L_f^n \hat{s}(\hat{x}) - L_f^n \hat{s}(\hat{y})\| \leqslant 2\delta^{-1} \|\hat{s}\|_\infty \,\hat{d}(\hat{x}, \hat{y}).$$

If $\hat{d}(\hat{x}, \hat{y}) \leqslant \delta$, then

$$\|L_f^n \hat{s}(\hat{x}) - L_f^n \hat{s}(\hat{y})\| \leqslant \sum_{\substack{c;\, o(c) = \hat{x} \\ |c| = n}} \{ |f(\pi(c))|\, \|\hat{s}(\ell(c) - \hat{s}(\ell(c'))\|$$
$$+ \|\hat{s}\|_\infty |f(\pi(c)) - f(\pi(c'))| \},$$

where c' denotes the companion of c with $o(c') = \hat{y}$. Note that

$$\|\hat{s}(\ell(c)) - \hat{s}(\ell(c'))\| \leqslant \mathrm{Lip}(\hat{s})\, \hat{d}(\ell(c), \ell(c'))$$
$$\leqslant \Lambda^n \,\mathrm{Lip}(\hat{s})\, \hat{d}(\hat{x}, \hat{y}). \qquad (3\text{-}1)$$

On the other hand, we observe that

$$\sum_{|c|=n} |f(\pi(c))-f(\pi(c'))| \leqslant \mathcal{N}\,\mathrm{Lip}(f)(1+\Lambda+\cdots+\Lambda^{n-1})\,\hat{d}(\hat{x},\hat{y})$$
$$\leqslant \mathcal{N}\,\mathrm{Lip}(f)(1-\Lambda)^{-1}\,\hat{d}(\hat{x},\hat{y}). \tag{3-2}$$

In fact this is proved by induction in the following way. For $n=1$, the assertion is obviously true. Supposing (3-2), we have

$$\sum_{\substack{|c|=n\\ e\in E:\, o(e)=t(c)}} |f(\pi(c\cdot e))-f(\pi(c'\cdot e'))|$$
$$\leqslant \sum |f(c)-f(c')|\,|f(e)| + \sum |f(c)|\,|f(e)-f(e')|$$
$$\leqslant \mathcal{N}\,\mathrm{Lip}(f)(1+\cdots+\Lambda^{n-1})\,\hat{d}(\hat{x},\hat{y}) + \mathcal{N}\Lambda^{n}\,\mathrm{Lip}(f)\,\hat{d}(\hat{x},\hat{y}),$$

where we have used the fact that $\sum |f(e)|=1$ and $\sum |f(c)|=1$. This proves that (3-2) holds for $n+1$. The lemma follows from (3-1) and (3-2).

Let $V=V_1\cup V_2\cup\cdots\cup V_\nu$ ($\nu=\nu(V,E,d)$) be the decomposition of V into the primitive parts. We then have an isomorphism

$$C^1(V)=\sum_{i=1}^{\nu}\oplus C^1(V_i).$$

From the definition of primitive decomposition, one may assume, permuting the indices $\{1,\ldots,\nu)$ if necessary, that $L_f(C^1(V_{i+1}))\subset C^1(V_i)$ for $i\in\mathbb{Z}/\nu\mathbb{Z}$. In particular, one has $L_f^{n\nu}(C^1(V_i))\subset C^1(V_i)$ for all $n\geqslant 1$.

LEMMA 3-5. *Suppose* $f\in C^1(E)$ *is positive valued and* $L_f 1=1$. *If* $g\in C^1(V_i)$ *satisfies*

$$\|L_f^{n\nu} g\|_\infty = \|g\|_\infty$$

for any $n\geqslant 1$, *then* $|g(x)|$ *and* $|L_f^{n\nu}g(x)|$, $n\geqslant 1$, *are constant on* V_i. *Furthermore, if* g *is real valued, then* g *is constant on* V_i.

Proof. Choose a sequence $\{x_n\}$ in V_i such that

$$|L_f^{n\nu}\,g(x_n)|=\|g\|_\infty.$$

Then we have

$$\|g\|_\infty \leqslant \sum_{\substack{c;\, o(c)=x_n\\ |c|=n\nu}} |f(c)|\,|g(t(c))| \leqslant \|g\|_\infty,$$

so $|g(x)| = \|g\|_\infty$ for x in $t(\bigcup_{n=1}^\infty \Omega_{x_n}(nv))$. In view of Lemma 2-6 in the previous section, we find that $|g(x)| = \|g\|_\infty$ on V_i. Further, since

$$\|L_f^{mv} L_f^{nv} g\|_\infty = \|g\|_\infty = \|L_f^{nv} g\|_\infty,$$

applying the same argument to $L_f^{nv} g$, we observe that $|L_f^{nv} g(x)|$ is constant on V_i.

Suppose now that g is real valued. In case $g(y_0) = \|g\|_\infty$ for some y_0 in V_i, one can find $x_n \in V_i$ such that $L_f^{nv} g(x_n) = \|g\|_\infty$, $n = 1, 2, \ldots$, since for a path c with $t(c) = y_0$, $|c| = nv$, one has $L_f^{nv} g(o(c)) > -\|g\|_\infty$, so $L_f^{nv} g(o(c)) = \|g\|_\infty$. In case $g(y_0) = -\|g\|_\infty$ for some $y_0 \in V_i$, we similarly find that $x_n \in V_i$ such that $L_f^{nv} g(x_n) = -\|g\|_\infty$. In any case, applying the same argument as above, we see that g is constant on V_i.

LEMMA 3-6. *Suppose that $f \in C^1(E)$ is positive valued and $L_f 1 = 1$. Then for any real valued $g \in C^1(V)$,*

(a) $\max\{g(x); x \in V_{i+1}\} \geqslant \max\{L_f g(x); x \in V_i\} \geqslant \min\{L_f g(x); x \in V_i\} \geqslant \min\{g(x); x \in V_{i+1}\}$,

(b) $\lim_{n\to\infty} L_f^{nv} g$ *exists in $\|\cdot\|_\infty$-topology, and is constant on each V_i.*

Proof. Property (a) is obvious from the definition. For (b), we may assume that g is real valued and belongs to $C^1(V_i)$ for some i. By Lemma 3-4, the sequence $\{L_f^{nv} g\}_{n=1}^\infty$ is uniformly bounded and equicontinuous. Choose a $\|\cdot\|_\infty$-convergent subsequence $\{L_f^{n_j v} g\}$, and set

$$g_0 = \lim_{j\to\infty} L_f^{n_j v} g.$$

For each $n \geqslant 1$, selecting an appropriate subsequence on $\{n_j\}$, we may assume that $n_k + n \leqslant n_{k+1}$. We then have

$$\begin{aligned}\|g_0\|_\infty \geqslant \|L_f^{nv} g_0\|_\infty &= \lim_{k\to\infty} \|L_f^{(n_k+n)v} g\|_\infty \geqslant \lim \|L_f^{n_{k+1} v} g\|_\infty \\ &= \|g_0\|_\infty,\end{aligned}$$

hence, by the above lemma, g_0 is constant on V_i. Since for every $n \geqslant n_j$

$$\begin{aligned}\|L_f^{nv} g - g_0\|_\infty &\leqslant \|L_f^{(n-n_j)v}(L_f^{n_j v} g - g_0) - (L_f^{n_j v} g - g_0)\|_\infty \\ &\quad + \|L_f^{n_j v} g - g_0\|_\infty \\ &\leqslant 3\|L_f^{n_j v} g - g_0\|_\infty,\end{aligned}$$

we conclude that $\lim_{n\to\infty} L_f^{nv} g = g_0$ in $\|\cdot\|_\infty$-topology.

We let C_0 be the finite dimensional subspace in $C^1(V)$ consisting of functions g_0 which are constant on each V_i. Let C_1 denote the kernel of the continuous linear operator:

$$\begin{aligned} C^1(V) &\to C_0 \\ g &\mapsto \lim_{n\to\infty} L_f^{n\nu} g = g_0. \end{aligned}$$

Then C_1 is closed and $C^1(V) = C_0 \oplus C_1$. It is also obvious that $L_f(C_0) \subset C_0$ and $L_f(C_1) \subset C_1$.

LEMMA 3-7. *Assume that* $L_f 1 = 1$. *Then we have*

(1) *The eigenvalues of* $L_f \mid C_0$ *are exactly* $\exp(2\pi\sqrt{-1}\,k/\nu)$, $k = 1, \ldots, \nu$.

(2) *The spectral radius of* $L_f \mid C_1$ *is strictly less than* 1.

Proof. Define functions $g_k \in C_0$, $k = 1, \ldots, \nu$, by setting

$$g_k(x) = \exp(2\pi\sqrt{-1}\,jk/\nu) \qquad \text{for} \quad x \text{ in } V_j.$$

Then $L_f g_k = \exp(2\pi\sqrt{-1}\,k/\nu)\, g_k$. Since $\dim C_0 = \nu$, we get (1).

To show (2), we let K denote the set of $g \in C_1$ with $\|g\|_1 = 1$, which is $\|\cdot\|_\infty$-compact. From Lemma 3-6(b), there exists n_0 such that for every $n \geq n_0$ and every $g \in K$

$$\|L_f^{n\nu} g\|_\infty < \min(1/4,\ 1/8C),$$

where C is the constant given in Lemma 3-4. On the other hand, we have

$$\begin{aligned} \mathrm{Lip}(L_f^{2n\nu} g) &\leq C\|L_f^{n\nu} g\|_\infty + \Lambda^{n\nu}\,\mathrm{Lip}(L_f^{n\nu} g) \\ &\leq C\|L_f^{n\nu} g\|_\infty + C\Lambda^{n\nu} + \Lambda^{2n\nu}, \end{aligned}$$

which is less than 1/4 for all $n \geq n_0$ provided that n_0 is taken large enough. Hence $\|L_f^{2n\nu}\|_1 < \frac{1}{2}$, and the spectral radius $r(L_f \mid C_1)$ is estimated as follows:

$$\begin{aligned} r(L_f \mid C_1) &= \underline{\lim}_{n\to\infty} \|L_f^n \mid C_1\|_1^{1/n} \\ &\leq \underline{\lim}_{n\to\infty} \|L_f^{2n\nu} \mid C_1\|_1^{1/2n\nu} \\ &\leq \left(\tfrac{1}{2}\right)^{1/2n_0\nu} < 1. \end{aligned}$$

This completes the proof.

Summarizing, we have the following theorem of Perron–Frobenius type.

THEOREM 3-8. *Let (V, E, d) be a compact irreducible topological graph with period ν. If $f \in C^1(E)$ is strictly positive, then*

(1) *the Ruelle operator $L_f\colon C^1(V) \to C^1(V)$ has a simple positive eigenvalue $\lambda(f)$ with a positive eigenfunction,*

(2) *$\lambda(f)\exp(2\pi\sqrt{-1}\,k/\nu)$, $k = 1,\dots,\nu$, are also simple eigenvalues.*

(3) *the rest of the spectrum contained in a disc of radius strictly less than $\lambda(f)$.*

The following lemma is easy to check.

LEMMA 3-9. *Under the same condition as in the above theorem,*

(i) *$\lambda(f) = \max\{\lambda;\ L_f g \geqslant \lambda g$ for some nonzero and non-negative function $g \in C^1(V)\}$,*

(ii) *if λ is a positive eigenvalue of L_f with a positive eigenfunction, then $\lambda = \lambda(f)$, and if $L_f g \geqslant \lambda(f)\, g$ for some non-negative function g, then $L_f g = \lambda(f)\, g$,*

(iii) *if we set $N_0 = \min_{x \in V} \#(o^{-1}(x))$ and $N^0 = \max_{x \in V} \#(o^{-1}(x))$, then*

$$N_0 \min(f) \leqslant \lambda(f) \leqslant N^0 \max(f),$$

(iv) *if $f' \geqslant f$, then $\lambda(f') \geqslant \lambda(f)$, and if $f' \geqslant f$ and $\lambda(f') = \lambda(f)$, then $f' = f$.*

LEMMA 3-10. *If $\lambda(1) = 1$, then (V, E) is a finite circuit graph.*

Proof. Let u be a positive eigenfunction of L_1 with the eigenvalue 1, and x_0 be a point in V such that $u(x_0) = \min u$. Then one has

$$\#(o^{-1}(x_0))\, u(x_0) \leqslant \sum_{e;\, o(e) = x_0} u(t(e)) = u(x_0).$$

This implies that $\#(o^{-1}(x_0)) = 1$ and $u(t(e)) = u(x_0)$ for $e \in o^{-1}(x_0)$. Thus the assumption of irreducibility leads to the conclusion that $u \equiv$ constant and $\#(o^{-1}(x)) \equiv 1$. Since (V, E) possesses at least one closed path (see Lemma 2-1), (V, E) must be isomorphic to a circuit graph.

The following lemma is required in proving that the poles of L-functions are simple (see next section).

LEMMA 3-11. *Let $l \in C^1(E)$ be positive valued. Then $(d/ds)\,\lambda(e^{-sl}) < 0$ for $s \in \mathbb{R}$.*

Proof. For brevity, we set $\varphi(s)=\lambda(e^{-sl})$ and $L_s=L_{\exp(-sl)}$. Suppose that $\varphi'(s_0)=0$ for some s_0. We can arrange for positive valued eigenfunction u_s of L_s with eigenvalue $\varphi(s)$ to depend smoothly on s and satisfy $(d/ds)\,u_s|_{s_0}>0$. Differentiating the both sides of the equation $L_s u_s=\varphi(s)\,u_s$ at $s=s_0$, we have

$$\left(\frac{d}{ds}L_s\bigg|_{s_0}\right)u_{s_0}+L_{s_0}\left(\frac{d}{ds}u_s\bigg|_{s_0}\right)=\varphi(s_0)\left(\frac{d}{ds}u_s\bigg|_{s_0}\right).$$

Since the value at $x\in V$ of the first term of the left hand side is equal to

$$-\sum_{e;\,o(e)=x} l(e)\,e^{-s_0 l(e)}u_{s_0}(t(e))<0,$$

we find that $L_{s_0}((d/ds)\,u_s|_{s_0})>\varphi(s_0)(d/ds)\,u_s|_{s_0}$, but this is impossible in view of (ii) in the above lemma.

Remark. Let $(\Sigma(V,E),\sigma)$ be a two-sided shift of finite type, that is

$$\Sigma(V,E)=\left\{\xi=(\xi^i)\in\prod_{-\infty}^{\infty}V;\,(\xi^i,\xi^{i+1})\in E\right\}$$

and σ is the shift operator of $\Sigma(V,E)$ given by $(\sigma\xi)^i=\xi^{i+1}$. Let l be a positive function on $\Sigma(V,E)$ which is Lipschitz continuous with respect to the distance;

$$d_\theta(\xi,\eta)=\theta^{\sup\{n;\,\xi^i=\eta^i\text{ for }|i|<n\}},\quad \theta\in(0,1).$$

It is known that l is cohomologous to a positive function l^+ depending only on the future $(\xi^i)_{i\geq 0}$, in the sense that there is a continuous ψ on $\Sigma(V,E)$ with $l=l^+-\psi+\psi\circ\sigma$. Regarding l^+ as a function on $\Sigma^+(V,E)$ (we may suppose that l^+ is Lipschitz continuous with respect to $d_{\theta'}$ for some $\theta'>\theta$), we take a positive h with $\lambda(\exp(-hl^+))=1$. It is known that the number h coincides with the topological entropy of the suspension flow $(\Sigma(V,E,l),\sigma(l)_t)$ associated with the function l. Here

$$\Sigma(V,E,l)=\{(\xi,t);\,\xi\in\Sigma(V,E),\,0\leqslant t\leqslant l(\xi)\}$$
$$\sigma(l)_t(\xi,s)=(\xi,s+t)$$

with appropriate identifications. Further

$$\frac{d}{ds}\lambda(e^{-sl^+})|_h=-\int l^+\,dm$$

where m is the equilibrium state for $-hl^+$.

4. Spectrum of Twisted Ruelle Operators

Let π: $(\hat{V}, \hat{E}, \hat{d}) \to (V, E, d)$ be a normal covering map with covering group G, and assume that (V, E, d) is compact, irreducible and has ν primitive parts. In this section, we intensively investigate the spectrum of the operator $L_{f,\rho}$ associated with a complex valued $f \in C^1(E)$. It is always assumed that f is nowhere vanishing. We easily verify that |spectrum of $L_{f,\rho}| \leqslant \lambda(|f|)$. We will make use of the following notions which are originally introduced by Pollicott [16] for functions on one-sided subshifts. Given a real $a \in [0, 2\pi)$, we say that f is an a-function with respect to ρ if $e^{\sqrt{-1}a}\lambda(|f|)$ is an eigenvalue of $L_{f,\rho}$. If f is *not* an a-function for any $a \in [0, 2\pi)$ then f is called *regular* with respect to ρ.

We first show that, if f is regular, the spectrum of $L_{f,\rho}$ is disjoint from the circle of radius $\lambda(|f|)$. As usual, we may assume that $L_{|f|}1 = 1$. We define an operator $T_{(a,n)}$ by setting

$$T_{(a,n)}s = \frac{1}{n}\sum_{k=0}^{n-1} (L_{\exp(\sqrt{-1}a)\cdot f,\rho})^k s, \qquad s \in C^1(F_\rho).$$

In view of the spectral mapping theorem, the set of spectrum of $T_{(a,n)}$ coincides with

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} \mu^k;\ \mu \text{ is a spectrum of } L_{\exp(\sqrt{-1}a)f,\rho}\right\}.$$

Lemma 4-1. *If f is regular with respect to ρ, then*

$$\lim_{n\to\infty} \|T_{(a,n)}s\|_\infty = 0$$

for every s.

Proof. It is immediate from the definition and Lemma 3-3 that $\|T_{(a,n)}s\|_\infty \leqslant \|s\|_\infty$ and $\mathrm{Lip}(T_{(a,n)}s) \leqslant C\|s\|_\infty + \mathrm{Lip}(s)$. This implies that $\{T_{(a,n)}s\}$ is uniformly bounded and equicontinuous, and that $\|T_{(a,n)}s\|_\infty$ must tend to zero since non-zero accumulation points of $\{T_{(a,n)}s\}$ would be eigenvectors for $L_{f,\rho}$ with eigenvalue $e^{-\sqrt{-1}a}$.

Proposition 4-2. *If f is regular with respect to ρ, then the spectral radius of $L_{f,\rho}$ is strictly less than $\lambda(|f|)$.*

Proof. We assume again that $L_{|f|}1 = 1$, and set $\Omega = \{s;\ \|s\|_1 \leqslant 1\}$, which is $\|\ \|_\infty$-compact. By the above lemma, for each positive ε, there exists n_ε such that if $n \geqslant n_\varepsilon$ then $\|T_{(a,n)}s\|_\infty < \varepsilon$ for $s \in \Omega$. Putting $m = [\sqrt{n}]$, we have

$$\begin{aligned}\mathrm{Lip}(T_{(a,n)}s) &\leqslant \mathrm{Lip}(T_{(a,n)}s - (L_{\exp(\sqrt{-1}a)f,\rho})^m \circ T_{(a,n)}s)\\ &\quad + \mathrm{Lip}((L_{\exp(\sqrt{-1}a)f,\rho})^m \circ T_{(a,n)}s)\\ &\leqslant \frac{2m}{n}\{C\|s\|_\infty + \mathrm{Lip}(s)\} + C\|T_{(a,n)}s\|_\infty\\ &\quad + \Lambda^m \,\mathrm{Lip}(T_{(a,n)}s),\end{aligned}$$

so that there is $n_0 \geqslant n_{1/4}$ such that if $n \geqslant n_0$ then $\mathrm{Lip}(T_{(a,n)}s) < \frac{1}{4}$ for $s \in \Omega$. Hence the spectral radius of $T_{(a,n)}$ is less than $\frac{1}{2}$ for large n. Thus 1 cannot be an element of the spectrum of $L_{\exp(\sqrt{-1}a)f,\rho}$ or equivalently $\exp(-\sqrt{-1}\,a)$ is not in the spectrum of $L_{f,\rho}$. Since a is arbitrary and the spectrum is closed we get the conclusion.

We shall see later that if a certain condition on connectedness of $(\hat{V}, \hat{E}, \hat{\partial})$ is satisfied, then f is regular with respect to all irreducible representations ρ with $\dim \rho > 1$.

We now investigate the spectra of $L_{f,\chi}$ for an a-function f with respect to a character χ.

PROPOSITION 4-3. *The following conditions are equivalent.*

(1) *f is an a-function with respect to χ.*

(2) *There exists a non-vanishing section $u \in C^1(F_\chi)$ such that the function $\hat{f}$ on $\hat{E}$ defined by*

$$\hat{f}(\hat{e}) = \exp(-\sqrt{-1}\,a)\, f(\pi(\hat{e}))\,\frac{u(t(\hat{e}))}{|u(t(\hat{e}))|}\,\frac{|u(o(\hat{e}))|}{u(o(\hat{e}))}$$

is positive valued and equals $|f|$. ($\hat{f}$ can be regarded as a function on $\hat{E}$.)

(3) *For any closed path c in (V, E),*

$$\chi(\langle c\rangle) = \exp(\sqrt{-1}\,|c|\,a)\,|f(c)|/f(c).$$

(Recall that we have defined $f(c)$ to be the product $f(e_1)\,f(e_2)\cdots f(e_n)$ for a path $c = (e_1,\dots, e_n)$.)

Proof. We may assume $L_{|f|}1 = 1$.

(1) $\Rightarrow$ (2). Let $u \in C^1(F_\chi)$ be an eigenfunction of $L_{f,\chi}$ with eigenvalue $\exp(\sqrt{-1}\,a)$ such that $\|u\|_\infty = 1$. We may regard $|u|$ as a function on V. Since $L_{|f|}|u| \geqslant |L_f u| = |u|$, we have that $|u| \equiv 1$ (see Lemma 3-9). If we take $\hat{f}$ with this u, then $L_{\hat{f}}1 = 1$. Since $L_{|\hat{f}|}1 = L_{|f|}1 = 1$, $\hat{f}$ turns out to be positive valued.

(2) ⇒ (3). Let c be a closed path in (V, E), and $\hat{c}$ be a lift of c. Then

$$|f(c)| = \hat{f}(\hat{c}) = \exp(-\sqrt{-1}\,|c|\,a)\, f(c)\, u(t(\hat{c}))\, u(o(\hat{c}))^{-1}$$
$$= f(c)\, \chi(\langle c \rangle) \exp(-\sqrt{-1}\,|c|\,a).$$

(3) ⇒ (1). We select an infinite path $c = (a_1, e_2, ...)$ such that $\{e_i\}$ is dense in E, and let $\hat{c} = (\hat{e}_1, \hat{e}_2, ...)$ be a lift of c. We denote

$$\hat{c}_{(n,m)} = (\hat{e}_{n+1}, ..., \hat{e}_m),$$
$$c_{(n,m)} = \pi(\hat{c}_{(n,m)}),$$

and set

$$u(\gamma, n) = \chi(\gamma) \exp(\sqrt{-1}\, na)\, f(c_{(0,n)})^{-1}\, |f(c_{(0,n)})|.$$

If $t(\gamma\hat{c}_{(0,n)}) = t(\mu\hat{c}_{(0,m)})$, $n < m$, then $c_{(n,m)}$ is closed and $\langle c_{(n,m)} \rangle = \gamma^{-1}\mu$, so that we have

$$\begin{aligned} u(\mu, m) &= \chi(\mu) \exp(\sqrt{-1}\, ma)\, f(c_{(0,m)})^{-1} |f(c_{(0,m)})| \\ &= \chi(\gamma) \exp(\sqrt{-1}\, na)\, f(c_{(0,n)})^{-1} |f(c_{(0,n)})| \\ &\quad \times \chi(\gamma^{-1}\mu) \exp(\sqrt{-1}\, |c_{(n,m)}|\, a)\, f(c_{(n,m)})^{-1} |f(c_{(n,m)})| \\ &= u(\gamma, n). \end{aligned}$$

This ensures that we can define a function u on the dense set

$$\hat{V}_0 = \{\gamma\hat{c}(n);\, n = 0, 1, 2, ... \quad \text{and} \quad \gamma \in G\}$$

by setting

$$u(\gamma\hat{c}(n)) = u(\gamma, n).$$

We claim that this function u can be extended to $\hat{V}$ as a Lipschitz continuous function and the resulting function is a non-zero eigenfunction of $L_{f,\chi}$ associated with $\exp(\sqrt{-1}\, a)$. For this, suppose that $\hat{d}(\gamma\hat{c}(n), \mu\hat{c}(m)) < \frac{1}{3}(1 - A) \min(\varepsilon, \delta)$, $n < m$. Then by Lemma 2-1, we can find a closed path c_∞ in (V, E) such that $|c_\infty| = m - n$ and $d(c(n+j), c_\infty(j)) < A^j(1 - A^{m-n})^{-1} \times d(c(n), c(m))$ $(< \frac{1}{3}\varepsilon)$, $0 \leqslant j \leqslant m - n$. We let $\hat{c}_\infty$ be a lift of c_∞ such that $\hat{d}(\hat{c}_\infty(0), \hat{c}(n)) = d(c_\infty(0), c(n))$. Then, in view of the properties of covering map,

$$\hat{d}(\hat{c}_\infty(j), \hat{c}(n+j)) = d(c_\infty(j), c(n+j)) \qquad \text{for} \quad 0 \leqslant j \leqslant m - n,$$

so that

$$\begin{aligned} \hat{d}(\hat{c}_\infty(m-n), \mu^{-1}\gamma\hat{c}_\infty(0)) \leqslant{}& \hat{d}(\hat{c}_\infty(m-n), \hat{c}(m)) + \hat{d}(\hat{c}(m), \mu^{-1}\gamma\hat{c}(n)) \\ &+ \hat{d}(\mu^{-1}\gamma\hat{c}(n), \mu^{-1}\gamma\hat{c}_\infty(0)). \end{aligned}$$

This implies that $\hat{c}_\infty(m-n)=\mu^{-1}\gamma\hat{c}_\infty(0)$ since $\pi(\hat{c}_\infty(m-n))=\pi(\hat{c}_\infty(0))$; hence $\langle c_\infty\rangle=\mu^{-1}\gamma$. We further have

$$
\begin{aligned}
&|u(\gamma\hat{c}(n))-u(\mu\hat{c}(m))|\\
&\quad=|1-\chi(\mu\gamma^{-1})\exp(\sqrt{-1}\,(m-n)\,a)\,f(c_{(n,m)})^{-1}|f(c_{(n,m)})||\\
&\quad=|f(c_\infty)^{-1}|f(c_\infty)|-f(c_{(n,m)})^{-1}|f(c_{(n,m)})||\\
&\quad\leqslant 2(1-\Lambda)^{-1}\operatorname{Lip}(f)(\inf|f|)^{-1}\,d(\gamma\hat{c}(n),\mu\hat{c}(m)),
\end{aligned}
$$

from which it follows that u has a Lipschitz extension to $\hat{V}$. which we also denote by u. From the definition of u, it is a trivial matter to check that $u(\gamma x)=\chi(\gamma)\,u(x)$; hence $u\in C^1(F_\chi)$. On the other hand, we find, again from the definition of u.

$$u(t(\hat{e}))\,f(\pi(\hat{e}))=\exp(\sqrt{-1}\,a)|f(\pi(\hat{e}))|\,u(o(\hat{e}))$$

for $\hat{e}\in\bigcup_{i=1}^{\infty}\bigcup_{\gamma\in G}\gamma\hat{e}_i$, and hence for $\hat{e}\in\hat{E}$. Summing up both sides over $\hat{e}$ with $o(\hat{e})=x$, we obtain

$$
\begin{aligned}
L_f u(x)&=\sum_{\hat{e}} f(\pi(\hat{e}))\,u(t(\hat{e}))=\exp(\sqrt{-1}\,a)\,u(x)\sum_{\hat{e}}|f(\pi(\hat{e}))|\\
&=\exp(\sqrt{-1}\,a)\,u(x).
\end{aligned}
$$

Thus u is an eigenfunction of $L_{f,\chi}$ associated with $\exp(\sqrt{-1}\,a)$, as desired.

COROLLARY 4-4. *If f is an a-function with respect to a character χ, then*

(1) $\exp(\sqrt{-1}\,(a+2\pi j\nu^{-1}))\cdot\lambda(|f|)$, $j=1,\ldots,\nu$, *are simple eigenvalues of $L_{f,\chi}$, where ν is the number of primitive parts of (V,E),*

(2) *the rest of the spectrum is contained in a disc with radius strictly less than $\lambda(|f|)$.*

Proof. Let u be a non-zero eigenfunction of $L_{f,\chi}$ with eigenvalue $\exp(\sqrt{-1}\,a)$. Note that the multiplication by u yields an isomorphism $C^1(V,\mathbb{C})\to C^1(F_\chi)$, and the diagram

$$
\begin{array}{ccc}
C^1(V,\mathbb{C}) & \xrightarrow{e^{\sqrt{-1}a}L_{f}} & C^1(V,\mathbb{C})\\
{\scriptstyle\times u}\downarrow & & \downarrow{\scriptstyle\times u}\\
C^1(F_\chi) & \xrightarrow{L_{f,\chi}} & C^1(F_\chi).
\end{array}
$$

is commutative. Thus our assertion reduces to Theorem 3-8.

We proceed to the case of an irreducible representation ρ with $\dim\rho\geqslant 2$. A topological graph (V,E,d) is called σ-connected if for every two points x

and y in (V, E) there are finitely many infinite step paths $c_1, c_2, \ldots, c_{2m}$ and integers $k_1, \ldots, k_m$ such that $o(c_1) = x$, $o(c_{2m}) = y$, $o(c_{2j}) = o(c_{2j+1})$ and $\lim_{k \to \infty} d(c_{2j-1}(k), c_{2j}(k + k_j)) = 0$ for $j = 1, \ldots, m$. In the case where (V, E, d) is a graph with discrete topology, this is equivalent to the usual connectedness of $|(V, E)|$ as a CW-complex.

PROPOSITION 4-5. *Let $\rho\colon G \to U(N)$ be an irreducible representation, and $N \geqslant 2$. If $(\hat{V}, \hat{E}, \hat{d})$ is σ-connected, every f is regular with respect to ρ.*

Proof. If this is not the case, then there exists non-zero $u \in C^1(F_\rho)$ satisfying $L_{f,\rho} u = e^{\sqrt{-1}a} u$ for some $a \in [0, 2\pi)$. (As usual we suppose $L_{|f|} 1 = 1$.) Then we find

$$L_{|f|} \|u\| \geqslant \|L_{f,\rho} u\| = \|u\|$$

hence $\|u\| \equiv$ constant in view of Lemma 3-9, so that we may suppose $\|u\| = 1$. Since

$$u(x) = \sum_{\substack{c;\, o(c) = x \\ |c| = n}} |f(\pi(c))| \frac{f(\pi(c))}{|f(\pi(c))|} \exp(-\sqrt{-1}\,|c|\, a)\, u(t(c)),$$

and $\sum |f(\pi(c))| = 1$, the convexity argument leads to the equality

$$\begin{aligned} u(t(c)) &= \exp(\sqrt{-1}\,|c|\, a) \frac{|f(\pi(c))|}{f(\pi(c))} u(o(c)) \\ &= \alpha(c)\, u(o(c)), \qquad |\alpha(c)| = 1. \end{aligned}$$

Let c and c' be two infinite paths in $(\hat{V}, \hat{E})$ such that $\lim_{j \to \infty} d(c(j), c'(j)) = 0$. Then

$$\begin{aligned} |\alpha(c_j)\, u(o(c_j)) - \alpha(c'_j)\, u(o(c'_j))| &= |u(t(c_j)) - u(t(c'_j))| \\ &\leqslant \mathrm{Lip}(u)\, d(c(j), c'(j)), \end{aligned}$$

where $c_j = c|[0, j]$. Since we may pick subsequences $\alpha(c_{j_k})$ and $\alpha(c'_{j_k})$ converging to some α and α', respectively, we get

$$\alpha u(o(c)) = \alpha' u(o(c')).$$

This implies, by virtue of σ-connectedness of $(\hat{V}, \hat{E})$, that for every two points x and y in $\hat{V}$ there is $\alpha(x, y) \in \mathbb{C}$ such that $|\alpha(x, y)| = 1$ and

$$u(y) = \alpha(x, y)\, u(x).$$

Especially, $\rho(\gamma)\,u(x)=u(\gamma x)=\alpha(\gamma x, x)\,u(x)$; hence the one dimensional subspace $\mathbb{C}u(x)$ is $\rho(G)$-invariant. This is a contradiction.

Summarizing, we have

THEOREM 4-6. *Let $\pi\colon (\hat{V}, \hat{E}, \hat{d}) \to (V, E, d)$ be a normal covering map of topological graph with covering transformation group G, and let $\rho\colon G \to U(N)$ be a unitary representation. We suppose that (V, E, d) is compact, irreducible and has period v, and that $(\hat{V}, \hat{E}, \hat{d})$ is σ-connected.*

(1) *If ρ is irreducible and $N \geqslant 2$, then every nowhere-vanishing $f \in C^1(E, \mathbb{C})$ is regular with respect to ρ.*

(2) *If ρ is a character and f is an a-function with respect to ρ, then* $\exp(\sqrt{-1}\,(a+2\pi j v^{-1}))\,\lambda(|f|)$, $j=1,..., v$, *are simple eigenvalues of $L_{f,\rho}$, and the rest of the spectrum is contained in a disc with radius strictly less than $\lambda(|f|)$.*

(3) *If f is regular with respect to ρ, then the spectrum of $L_{f,\rho}$ is contained in a disc with radius strictly less than $\lambda(|f|)$.*

We conclude this section by giving an example of normal covering map with σ-connected total space. Let $(\varprojlim V_n, \varprojlim E_n)$ be an irreducible profinite graph, and let $\pi_1\colon (\hat{V}_1, \hat{E}_1) \to (V_1, E_1)$ be the universal covering. For each $n > 1$, one can construct a normal covering map $\pi_n\colon (\hat{V}_n, \hat{E}_n) \to (V_n, E_n)$ whose covering transformation group is $\pi_1(V_1, E_1)$, and a morphism $\hat{\omega}_n\colon (\hat{V}_n, \hat{E}_n) \to (\hat{V}_{n-1}, \hat{E}_{n-1})$ which is a lift of ω_n and commutes with the $\pi_1(V_1, E_1)$-action. Indeed, it suffices to define $(\hat{V}_n, \hat{E}_n)$ to be the quotient graph of the universal covering of (V_n, E_n) by the action of $\mathrm{Ker}(\omega_2 \circ \cdots \circ \omega_n)_*$ $(\subset \pi_1(V_n, E_n))$. The covering transformation group of π_n is

$$\pi_1(V_n, E_n)/\mathrm{Ker}(\omega_2 \circ \cdots \circ \omega_n)_* \simeq \pi_1(V_1, E_1),$$

since the homomorphism $(\omega_n)_*\colon \pi_1(V_n, E_n) \to \pi_1(V_{n-1}, E_{n-1})$ is surjective (note that $\pi_1(V_n, E_n)$ is generated by closed paths). We then observe that the sequence $(\hat{V}_n, \hat{E}_n)_{n \geqslant 1}$ and morphisms $\hat{\omega}_n$ satisfy the conditions (PLG1)–(PLG4) and the morphism

$$\pi = \varprojlim \pi_n\colon (\varprojlim \hat{V}_n, \varprojlim \hat{E}_n) \to (\varprojlim V_n, \varprojlim E_n)$$

is a normal covering map whose covering transformation group is $\pi_1(V_1, E_1)$. In fact, for every $\hat{x}$ in $\hat{V}_{n-1}$, one can find $\hat{y}$ in $\hat{V}_n$ with $\pi_{n-1}(\hat{\omega}_n(\hat{y})) = \omega_n(\pi_n(\hat{y})) = \pi_{n-1}(\hat{x})$, so that $\hat{\omega}_n(\hat{y}) = \gamma\hat{x}$ for some $\gamma \in \pi_1(V_1, E_1)$, and $\hat{\omega}_n(\gamma^{-1}\hat{y}) = \gamma^{-1}\hat{\omega}_n(\hat{y}) = \hat{x}$. Hence we have (PLG1). Condition (PLG2) comes from that each π_n is a covering map and $\{(V_n, E_n)\}$ satisfies (PLG2). For (PLG3), suppose that $\hat{\omega}_n(\hat{e}_1) = \hat{\omega}_n(\hat{e}_2)$. Then

$\omega_n(\pi_n(\hat{e}_1)) = \omega_n(\pi_n(\hat{e}_2))$; hence $\pi_n(\ell(\hat{e}_1)) = \pi_n(\ell(\hat{e}_2))$ and $\ell(\hat{e}_1) = \gamma\ell(\hat{e}_2)$ for some $\gamma \in \pi_1(V_1, E_1)$. But $\hat{\omega}_n(\ell(\hat{e}_2)) = \hat{\omega}_n(\ell(\hat{e}_1)) = \gamma\hat{\omega}_n(\ell(\hat{e}_2))$, so that $\gamma = 1$ (the neutral element of G) and $\ell(\hat{e}_1) = \ell(\hat{e}_2)$. Condition (PLG4) is obvious. Since each π_n is a covering map of finite graphs, the condition (C-1) is immediate. To see that π is a local isometry, suppose that $\hat{u}_{n0} = \hat{v}_{n0}$, $\hat{u}_{n0+1} \neq \hat{v}_{n0+1}$ for $\hat{u} = (\hat{u}_n)$ and $\hat{v} = (\hat{v}_n)$ in $\varprojlim \hat{V}_n$. If $\pi_{n0+1}(\hat{u}_{n0+1}) = \pi_{n0+1}(\hat{v}_{n0+1})$, then $\hat{u}_{n0+1} = \gamma\hat{v}_{n0+1}$ for some γ; hence $\hat{u}_{n0} = \gamma\hat{v}_{n0}$, and $\gamma = 1$. But this is again a contradiction. In terms of the metric d_θ, this means that if $d_\theta(u, v) \leqslant \theta$, then $d_\theta(\pi(\hat{u}), \pi(\hat{v})) = d_\theta(\hat{u}, \hat{v})$. On the other hand, if $d_\theta(u, v) \leqslant \theta$, then one can find some $\hat{u}, \hat{v}$ such that $d_\theta(\pi(\hat{u}), \pi(\hat{v})) = d_\theta(\hat{u}, \hat{v})$, whence (C-2).

To see that $(\varprojlim \hat{V}_n, \varprojlim \hat{E}_n)$ is σ-connected, it suffices to show that a general projective limit of graphs $(V, E, d_\theta) = \varprojlim(V_n, E_n)$ satisfying the conditions (PLG1)–(PLG4) is σ-connected provided that (V_1, E_1) is connected as a CW-complex. For this, it is enough to prove that for $u = (u_n)$ and $v = (v_n)$ with $u_1 = v_1$, there exist infinite paths c and c' satisfying

$$o(c) = u,\ o(c') = v \qquad \text{and} \qquad \lim_{j \to \infty} d_\theta(c(j), c'(j)) = 0.$$

Let c_1 be an infinite path in (V_1, E_1) with $o(c_1) = u_1 = v_1$. In view of the condition (PLG2), one can find paths c and c'such that $\tilde{\omega}_n(c) = c_1$, $o(c) = u$, $\tilde{\omega}_n(c') = c_1$, $o(c') = v$. The condition (PLG3) guarantees that $\lim d_\theta(c(j), c'(j)) = 0$.

5. L-Functions of Pro-finite Graphs

Let $(V, E, d_\theta) = (\varprojlim V_n, \varprojlim E_n, d_\theta)$ be a pro-finite irreducible graph with ν primitive parts, and let l be a Lipschitz continuous length function on E. It is assumed throughout this section that (V, E) is not a circuit graph. We set $M = \sup_{x \in V} \# o^{-1}(x)$. As usual $\tilde{\omega}_n$: $(V, E) \to (V_n, E_n)$ denotes the canonical projection. The fundamental group of (V, E) is defined to be the projective limit

$$\pi_1(V, E) = \varprojlim \pi_1(V_n, E_n).$$

Let ρ: $\pi_1(V, E) \to U(N)$ be a continuous homomorphism, and $L(s, \rho)$ be the L-function associated to ρ. The goal of this section is to prove Theorem D.

It is a standard fact that ρ can be factored as

$$\pi_1(V, E) \longrightarrow \pi_1(V_{n0}, E_{n0}) \xrightarrow{\rho_{n_0}} U(N)$$

for some $n_0 \geqslant 1$. We may assume without loss of generality that $n_0 = 1$. For simplicity, we write $\rho = \rho_1$ (by abuse of notation). Let π: $(V, E, d_\theta) \to$

(V, E, d_θ) be the normal covering map constructed in the end of the previous section. We denote by F_ρ the flat vector bundle on V associated with the representation ρ of the covering transformation group $G = \pi_1(V_1, E_1)$. Since we freely vary the parameter θ, we shall make use of the following notations in order to express explicitly the dependence on θ.

$$\mathrm{Lip}_\theta(\cdot) = \text{the Lipschitz constant with respect to } d_\theta,$$

$$\|\cdot\|_\theta = \|\cdot\|_1\text{-norm with respect to } d_\theta.$$

$$C_\theta(E) = C^1(E),$$

$$B_\varepsilon^\theta(f) = \{f' \in C_\theta^1(E);\ \|f' - f\|_\theta < \varepsilon\}, \text{ and so on.}$$

It is a trivial matter to check that $C_\theta^1 \subset C_{\theta'}^1$ for $\theta \leqslant \theta'$. Denote by $\Gamma_n(\rho)$ the finite dimensional subspace of $C_\theta^1(F_\rho)$ consisting of functions of the form $g \circ \hat{\tilde{\omega}}_n$, where $\hat{\tilde{\omega}}_n$: $(\hat{V}, \hat{E}) \to (\hat{V}_n, \hat{E}_n)$ denotes the projection, and g are $\mathbb{C}^N$-valued functions on $\hat{V}_n$ satisfying $g(\gamma x) = \rho(\gamma)\, g(x)$ for $\gamma \in G$ and $x \in \hat{V}_n$. One checks without difficulty that $L_{f \cdot \tilde{\omega}_n}(g \circ \hat{\tilde{\omega}}_n) = (L_f g) \circ \hat{\tilde{\omega}}_n$ for every function f on E_n, so that $L_{f \cdot \tilde{\omega}_n, \rho}(\Gamma_n(\rho)) \subset \Gamma_n(\rho)$. In fact this comes from the condition (PLG2). Given a function $f \in C_\theta^1(E)$, we define a function f_n on E_n by setting

$$f_n(e) = \sup_{\tilde{\omega}_n(\mathbf{e}) = e} \mathrm{Re}\, f(\mathbf{e}) + \sqrt{-1} \sup_{\tilde{\omega}_n(\mathbf{e}) = e} \mathrm{Im}\, f(\mathbf{e}).$$

We similarly define g_n for a function $g \in C_\theta^1(V)$.

We have the following useful lemma, which is easy to show.

LEMMA 5-1. (1) $\|f_n \circ \tilde{\omega}_n - f\|_\infty \leqslant \mathrm{Lip}_\theta(f)\, \theta^n$.

(2) $\mathrm{Lip}_{\theta'}(f_n \circ \tilde{\omega}_n - f) \leqslant 2\, \mathrm{Lip}_\theta(f)(\theta/\theta')^n$, $0 < \theta < \theta' < 1$.

The same estimates hold for $g \in C_\theta^1(V)$.

LEMMA 5-2. *For a positive valued* $f \in C_\theta^1(E)$,

$$\log \lambda(f) = \overline{\lim_{m \to \infty}} \frac{1}{m} \log \Bigg[\sum_{\substack{c;\text{closed} \\ |c| = m}} f(c) \Bigg],$$

where $\lambda(f)$ *is the maximum positive eigenvalue of* L_f.

Proof. It follows from Lemma 2-2 that the right hand side is finite. The above lemma, Lemma 3-2 and Theorem 3-8 guarantee that

$$\lim_{n \to \infty} \lambda(f_n \circ \tilde{\omega}_n) = \lambda(f). \tag{5-1}$$

On the other hand, due to Lemma 3-9(ii), we have

$$\lambda(f_n \circ \tilde{\omega}_n) = \lambda(f_n) = \lim_{m \to \infty} \{\operatorname{tr} L_{f_n}^m\}^{1/m}$$
$$= \lim_{m \to \infty} \left\{\sum_c f_n(c)\right\}^{1/m}$$

(cf. the argument in Section 1), where, in the summation, c runs over close paths in (V_n, E_n) with $|c| = m$. But the last quantity equals

$$\lim_{m \to \infty} \left\{\sum_c f_n(\tilde{\omega}_n c)\right\}^{1/m},$$

where, in this turn, c runs over closed paths in (V, E) with $|c| = m$. Hence it suffices to show that

$$\lim_{n \to \infty} \left[\varlimsup_{m \to \infty} \frac{1}{m} \log \left\{ \sum_{|c| = m} f_n(\tilde{\omega}_n(c)) \Big/ \sum_{|c| = m} f(c) \right\}\right] = 0.$$

Note that

$$f_n(\tilde{\omega}_n(c))/f(c) = \left(1 + \frac{f_n(\tilde{\omega}_n(e_1)) - f(e_1)}{f(e_1)}\right) \times \cdots$$
$$\times \left(1 + \frac{f_n(\tilde{\omega}_n(e_m)) - f(e_n)}{f(e_m)}\right)$$
$$\leqslant \left(1 + \frac{\operatorname{Lip}_\theta(f)}{\inf(f)}\,\theta^n\right)^m,$$

where $c = (e_1, ..., e_m)$. Hence

$$\varlimsup_{m \to \infty} \frac{1}{m} \log \left[\sum f_n(\tilde{\omega}_n(c) / \sum f(c)\right] \leqslant \log \left(1 + \frac{\operatorname{Lip}_\theta(f)}{\inf(f)}\,\theta^n\right).$$

This completes the proof.

Though our concern is the analyticity of the L-function $L(s, \rho)$, we shall briefly treat the general function $L(f, \rho)$ defined for $f \in C_\theta^1(E)$:

$$L(f, \rho) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{c;\text{closed paths}\\ \text{with } |c| = m}} \operatorname{tr} \rho(\langle c \rangle) f(c)\right).$$

We call $L(f, \rho)$ the L-function on $C_\theta^1(E)$ associated with the representation ρ. Apparently $L(s, \rho) = L(e^{-st}, \rho)$ (see Section 1). We are interested in the meromorphic domain of $L(f, \rho)$. For this we set

$$U_\theta = \{f \in C^1_\theta(E, \mathbb{C});\ |f| > 0,\ \lambda(|f|) < 1\},$$
$$\partial U_\theta = \{f \in C^1_\theta(E, \mathbb{C});\ |f| > 0,\ \lambda(|f|) = 1\}.$$

Lemma 5-2 implies that $L(f, \rho)$ converges and is analytic for $f \in U_\theta$. If we let h be a positive number such that $\lambda(e^{-hl}) = 1$ (which is uniquely determined in view of Lemma 3-9 or Lemma 3-10, we then observe

$$e^{-sl} \in U_\theta \ \leftrightarrow \mathrm{Re}\, s > h$$
$$e^{-sl} \in \partial U_\theta \leftrightarrow \mathrm{Re}\, s = h$$

so that $L(s, \rho)$ is holomorphic in $\mathrm{Re}\, s > h$.

LEMMA 5-3. *If $f \in \partial U_\theta$ is regular with respect to ρ, then $L(\cdot, \rho)$ is analytic and non-zero in a neighborhood of f. In particular, this is the case if ρ is irreducible and* $\dim \rho \geqslant 2$.

Proof. From the assumption, there exists β such that $r(L_{f,\rho}) < \beta < 1$, where $r(L_{f,\rho})$ denotes the spectral radius of the operator $L_{f,\rho}$. Choose $\theta' > \theta$ and set

$$\alpha = -\,[\text{illegible}]^{-1} \log \beta.$$

Since the function $f' \mapsto r(L_{f',\rho})$ is upper semicontinuous and

$$\|f'_n \circ \tilde{\omega}_n - f\|_{\theta'} \leqslant 2\|f'\|_\theta (\theta/\theta')^n + \|f' - f\|_{\theta'},$$

there exists a positve $\varepsilon < \min|f|$ and an integer n_1 such that $r(L_{f'_n \circ \tilde{\omega}_n, \rho}) < \beta$ and $\lambda(|f'|) < \theta^{-\alpha/2}$ for $f' \in B^\theta_\varepsilon(f)$ and $n \geqslant n_1$.

We set $n(m) = [m\alpha] - 1$, and take m_1 so that $n(m_1) \geqslant n_1$. If we put $R' = (\min|f'|)^{-1} \|f'\|_\theta$, then we have, for every closed path c in (V, E) with $|c| = m$,

$$|f'(c) - f'_n \circ \tilde{\omega}_n(c)| \leqslant |f'(c)|\ m\{1 + R'\theta^n\}^m R'\theta^n.$$

When $f' \in B^\theta_\varepsilon(f)$,

$$R' \leqslant (\|f\|_\theta + \varepsilon)(\min|f| - \varepsilon)^{-1}$$

and

$$\varlimsup_{m \to \infty} \left| \frac{1}{m} \sum_{c;|c| = m} \mathrm{tr}\, \rho(\langle c \rangle)\{f'(c) - f'_{n(m)} \circ \tilde{\omega}_{n(m)}(c)\} \right|^{1/m}$$
$$\leqslant \varlimsup_{m \to \infty} N^{1/m}\{1 + R'\theta^{n(m)}\}\ R'^{1/m}\theta^\alpha \left\{\sum_c |f'(c)|\right\}^{1/m}$$
$$\leqslant \theta^{\alpha/2} < 1.$$

This implies that the sum

$$\sum_{m=m_1}^{\infty} \frac{1}{m} \sum_{c,|c|=m} \operatorname{tr} \rho(\langle c \rangle)\{f'(c) - f'_{n(m)} \circ \tilde{\omega}_{n(m)}(c)\}$$

converges absolutely and uniformly on $B_\varepsilon^\theta(f)$.

On the other hand, if m is sufficiently large, then for every $f' \in B_\varepsilon^\theta(f)$, we have

$$\begin{aligned}
&\left| \frac{1}{m} \sum_{c:|c|=m} \operatorname{tr} \rho(\langle c \rangle) f'_{n(m)} \circ \tilde{\omega}_{n(m)}(c) \right|^{1/m} \\
&\quad = \left| \frac{1}{m} \operatorname{tr} (L_{f'_{n(m)}, \rho})^m \right|^{1/m} \leqslant \beta(\#(V_{n(m)}))^{1/m} N^{1/m} \\
&\quad \leqslant \beta \cdot 1^{n(m)/m} \#(V_1)^{1/m} N^{1/m} \qquad \text{(see (2-1))} \\
&\quad \leqslant \beta \cdot 1^{\alpha} \#(V_1)^{1/m} N^{1/m} < \beta' < 1,
\end{aligned}$$

hence the sum $\sum_{m=m_1}^{\infty} (1/m) \sum_{|c|=m} \operatorname{tr} \rho(\langle c \rangle)\, f'_{n(m)} \circ \tilde{\omega}_{n(m)}(c)$ converges absolutely and uniformly on $B_\varepsilon^\theta(f)$. This completes the proof.

We now focus our attention on the case of irregular function $f \in \partial U_\theta$ with respect to a character χ. We first show

LEMMA 5-4. *If $f \in \partial U_0$ is irregular with respect to χ, then $L(\cdot, \chi)$ has meromorphic extension to a neighborhood of f.*

Proof. Assume that f is an a-function with respect to χ. Consider the correspondence

$$f' \mapsto g' = f'|f|/f.$$

By Proposition 4-3, we find $L(f', \chi) = L(e^{\sqrt{-1}a} g', \mathbb{1})$. This implies that the assertion reduces to the analyticity of $L(e^{\sqrt{-1}a} f', \mathbb{1})$ with respect to $f' \in B_\varepsilon(f)$, where f is a positive function with $\lambda(f) = 1$. By analyticity of perturbed simple eigenvalues and associated eigenvectors, one can find holomorphic functions $\lambda_j(f')$ defined on $B_\varepsilon(f)$ satisfying

$$\begin{aligned}
\lambda_j(f) &= \exp(2\pi \sqrt{-1}\, j/\nu), \qquad j = 0, ..., \nu - 1, \\
L_{f'} u_j(f') &= \lambda_j(f')\, u_j(f'),
\end{aligned}$$

where $u_j(f') \in C_\theta^1(V)$ depends continuously on f' and $\|u_j(f')\|_\theta = 1$. One may assume that $\beta < |\lambda_j(f')| < 2$ and the rest of the spectrum of $L_{f'}$ is contained in a disc of radius less than β for some positive $\beta < 1$. Furthermore, taking sufficiently small ε, we may assume $\lambda(|f'|) < \theta^{-\alpha/2}$ and $\|u_j(f') -$

$u_j(f)\|_\theta < \frac{1}{2} \min |u_j(f)| \leqslant \frac{1}{2}$, where $\alpha = -M^{-1} \log \beta$. Define functions $u_n^j(f')$ and $v_n^j(f') \in \Gamma_n(\mathbb{1})$ by

$$\begin{aligned} u_n^j(f') &= L_{f'_n}(u_j(f'))_n \circ \tilde{\omega}_n - \lambda_j(f')\, u_j(f')_n \circ \tilde{\omega}_n, \\ v_n^j(f') &= \lambda_j(f')^{-1} L_{f'_n}(u_j(f'))_n \circ \tilde{\omega}_n \\ &\quad (= u_j(f')_n \circ \tilde{\omega}_n - \lambda_j(f')^{-1} u_n^j(f')). \end{aligned}$$

Then we have

$$\begin{aligned} \|u_n^j(f')\|_\infty &\leqslant \|L_{f'_n \circ \tilde{\omega}_n}(u_j(f')_n \circ \tilde{\omega}_n) - L_{f'}(u_j(f') \circ \tilde{\omega}_n)\|_\infty \\ &\quad + \|L_{f'}\{u_j(f')_n \circ \tilde{\omega}_n - u_j(f')\}\|_\infty \\ &\quad + |\lambda_j(f')|\, \|u_j(f') - u_j(f')_n \circ \tilde{\omega}_n\|_\infty \\ &\leqslant M \|u_j(f')\|_\infty \|f'_n \circ \tilde{\omega}_n - f'\|_\infty \\ &\quad + \{M \|f'\|_\infty + 2\}\, \|u_j(f') - u_j(f')_n \circ \tilde{\omega}_n\|_\infty \\ &\leqslant 2M \{\|f\|_\theta + \varepsilon + 1\}\, \theta^n, \end{aligned}$$

and

$$\begin{aligned} \min |v_n^j(f')| &\geqslant \min |u_j(f')| - |\lambda_j(f')|\, \|u_n^j(f')\|_\infty \\ &\geqslant \tfrac{1}{2} \min |u_j(f)| - 4M \{\|f\|_\theta + \varepsilon + 1\}\, \theta^n. \end{aligned}$$

Hence there is a positive integer n_1 such that $\min |v_n^j(f')| \geqslant \frac{1}{4} \min |u_j(f)|$ for all $n \geqslant n_1$, and

$$\begin{aligned} &\|u_j(f')_n (v_n^j(f'))^{-1} - 1\|_\infty \\ &\quad \leqslant 4 |\lambda_j(f')|^{-1} \|u_n^j(f')\|_\infty (\min |u_j(f)|)^{-1} \\ &\quad \leqslant C\theta^n, \end{aligned} \tag{5-2}$$

$$\begin{aligned} &\mathrm{Lip}_{\theta'}\{u_j(f')_n - v_n^j(f')\} \\ &\quad \leqslant \mathrm{Lip}_{\theta'}\{u_j(f')_n - u_j(f')\} + |\lambda(f')|^{-1}\, \mathrm{Lip}_{\theta'}\{L_{f'} u_j(f') - L_{f'_n} u_j(f')_n\} \\ &\quad \leqslant C(\theta/\theta')^n \end{aligned} \tag{5.3}$$

for some positive constant C. We now define $\bar{f}'_n \colon E \to \mathbb{C}$ by

$$\bar{f}'_n(e) = f'_n \circ \tilde{\omega}_n(e) \{u_j(f')_n(\ell(e))\} \{v_n^j(f')(\ell(e))\}^{-1}.$$

From definition of $u_j(f')_n$ and $v_n^j(f')$, the function $\bar{f}'_n$ has the form $g'_n \circ \tilde{\omega}_n$

for some function g'_n on E_n. Combining (5-2) with Lemma 5-1, we easily have

$$\|\bar{f}'_n - f'\|_\infty \leqslant C'\theta^n$$
$$\|\bar{f}'_n - f'\|_{\theta'} \leqslant C'(\theta/\theta')^n, \qquad \text{for some} \quad C' > 0.$$

Since we have

$$L_{\bar{f}'_n} v_n^j(f') = \lambda_j(f')\, v_n^j(f'),$$

it follows that there exists an integer n_1 such that $\lambda_j(\bar{f}'_n) = \lambda_j(f')$ for $f' \in B_{\varepsilon/2}(f)$ and $n > n_1$.

We set $n(m) = [m\alpha] - 1$, and choose m_1 so that $n(m_1) > n_1$. Employing the same argument as in Lemma 5-3, we deduce that

$$\begin{aligned}
&\varlimsup_{m\to\infty} \left| \frac{1}{m} \sum_{|c|=m} \{f'(c) - \bar{f}'_{n(m)}(c)\} \right|^{1/m} \\
&\quad \leqslant \varlimsup \{1 + R''\theta^{n(m)}\}\, R''^{1/m}\theta^\alpha \left\{ \sum_{|c|=m} |f'(c)| \right\}^{1/m} \\
&\quad < \theta^{\alpha/2} < 1,
\end{aligned}$$

where $R'' = C'/\min|f'|$. Hence the sum

$$\sum_{m=m_1}^\infty \frac{e^{\sqrt{-1}m a}}{m} \sum_{|c|=m} \{f'(c) - \bar{f}'_{n(m)}(c)\}$$

converges absolutely and uniformly for $f' \in B^\theta_{\varepsilon/2}(f)$. On the other hand, we observe that

$$\begin{aligned}
&\left| \frac{1}{m} \left\{ \sum_{|c|=m} \bar{f}'_{n(m)}(c) - \sum_{j=1}^{v} \lambda_j(f')^m \right\} \right|^{1/m} \\
&\quad \leqslant \left| \operatorname{tr}(L_{g'_{n(m)}})^m - \sum_{j=1}^{v} \lambda_j(g'_{n(m)})^m \right|^{1/m} \\
&\quad \leqslant \beta(A^{n(m)} \# (V_1))^{1/m} \leqslant \beta A^\alpha (\#(V_1))^{1/m}.
\end{aligned}$$

This leads us to the conclusion that

$$\sum_{m=m_1}^\infty \frac{e^{\sqrt{-1}m a}}{m} \left\{ \sum_{|c|=m} \bar{f}'_{n(m)}(c) - \sum_{j=1}^{v} \lambda_j(f')^m \right\}$$

converges absolutely and uniformly on $B^\theta_\varepsilon(f)$. This, in conjunction with the above argument, shows that, for every $f' \in B^{\theta}_{\varepsilon/2}(f)$ with $\lambda_j(f') \neq e^{-\sqrt{-1}a}$, $j = 0, ..., \nu - 1$,

$$
\begin{aligned}
L(e^{\sqrt{-1}a}f', \mathbb{1})
&= \prod_{j=1}^{\nu} (1 - e^{\sqrt{-1}a}\lambda_j(f'))^{-1} \\
&\times \exp \sum_{m=1}^{m_1 - 1} \frac{e^{\sqrt{-1}ma}}{m} \sum_{|c| = m} f'(c) \\
&\times \exp \sum_{m=m_1}^{\infty} \frac{e^{\sqrt{-1}ma}}{m} \sum_{|c| = m} \{f'(c) - \bar{f}'_{n(m)}(c)\} \\
&\times \exp \sum_{m=m_1}^{\infty} \frac{e^{\sqrt{-1}am}}{m} \left\{ \sum_{|c|=m} \bar{f}'_{n(m)}(c) - \sum_{j=1}^{\nu} \lambda_j(f') \right\},
\end{aligned}
$$

where the summations in the exponential function converge absolutely and uniformly in a neighborhood of f. This completes the proof.

Summarizng, we obtain

THEOREM 5-5. *The L-function $L(f, \rho)$ is holomorphic in $U_\theta = \{f \in C^1_\theta(E);$ $|f| > 0,\ \lambda(|f|) < 1\}$ and meromorphically continued to an open domain containing $\partial U_\theta = \{f;\ |f| > 0,\ \lambda(|f|) = 1\}$, on which $L(f, \rho)$ does not have zero. If ρ is irreducible and* $\dim \rho \geq 2$, *then $L(f, \rho)$ is holomorphic on ∂U_θ. For a character χ, $f \in \partial U_\theta$ is a pole of $L(f, \chi)$ if and only if f is a 0-function with respect to χ, that is, 1 is an eigenvalue of $L_{f,\chi}$. This being the case, χ must satisfy the relation*

$$\chi(\langle c \rangle) = |f(c)|/f(c), \qquad \text{for any closed path } c,$$

and $L(f', \chi) = L(f'|f|/f, \mathbb{1})$ for f' around f. Moreover, the function $f' \mapsto (1 - \lambda(f'(|f|/f)))\, L(f', \chi)$ is holomorphic around f.

Theorem D is an immediate consequence of this theorem and Lemma 3-11.

6. *L*-FUNCTIONS OF ANOSOV FLOWS

This section is devoted to showing Theorems A and B.

Let X be a Riemannian manifold. A smooth flow ϕ_t on X is called of *Anosov type* if the following condition holds: There exists a ϕ_t-invariant splitting of the tangent bundle of X

$$TX = E^{\mathrm{T}} \oplus E^{\mathrm{s}} \oplus E^{\mathrm{u}},$$

where E^T is the line bundle tangent to the orbits of the flow and E^s and E^u are exponentially contracting and expanding respectively; for some constants $C>0$, $\lambda>0$

$$\|d\phi_t(v)\| \leqslant Ce^{-\lambda t}\|v\|, \qquad v \in E^s,\ t \geqslant 0,$$

$$\|d\phi_{-t}(v)\| \leqslant Ce^{-\lambda t}\|v\|, \qquad v \in E^u,\ t \geqslant 0.$$

The following proposition is a generalization of Bowen's result [2], which gives a relation between the L-function of an Anosov flow and those of certain topological graphs attached to the flow.

PROPOSITION 6-1. *There is a family of irreducible pro-finite graphs* $(\mathbf{V}_a, \mathbf{E}_a\ d_a)$ *with Lipschitz continuous length functions* l_a^+, $a=0, 1,\dots, \mu$, *and continuous homomorphisms* $\varphi_a\colon \pi_1(\mathbf{V}_a, \mathbf{E}_a) \to \pi_1(X)$ *such that for any unitary representation* ρ *of* $\pi_1(X)$

$$L(s, \rho) = L(s, \rho \cdot \varphi_0) \prod_{a=1}^{\mu} L(s, \rho \cdot \varphi_a)^{(-1)^{q(a)}}$$

and that the topological entropy h *of* ϕ_t *equals* h_0 *and* $h_a < h_0$ *for* $a>0$, *where* $q(a)$ *are integers associated to the graphs* $(\mathbf{V}_a, \mathbf{E}_a)$ *and* h_a *are positive numbers determined by the equation* $\lambda(\exp(-h_a l_a^+))=1$. *In particular, the* L-*function* $L(s, \rho)$ *has a nowhere vanishing meromorphic extension to an open neighborhood of* $\operatorname{Re} s \geqslant h$.

Proof. Let V_0 be a Markov family for the flow (X, ϕ_t), which is, roughly speaking, a finite family of disjoint local cross sections to the flow ϕ_t. If one defines $E_0 \subset V_0 \times V_0$ to be the set of all (v, w) such that some "interior" point in v goes firstly into w along ϕ_t (see [2] for precise definitions), then from recurrence property, it follows that the oriented finite graph (V_0, E_0) is irreducible. By a fundamental theorem established by Bowen, one can find a positive valued d_θ-Lipschitz continuous function l_0 on $\Sigma(V_0, E_0)$ and a continuous finite to one surjective map τ: $\Sigma(V_0, E_0, l_0) \to X$ such that

(1) $\tau \circ \sigma(l_0)_t = \phi_t \circ \tau$,

(2) $\tau(\xi, 0) \in \xi^0$ for all $\xi \in \Sigma(V_0, E_0)$,

(3) the topological entropy h of (X, ϕ_t) coincides with that of the suspension flow $(\sum(V_0, E_0, l_0),\ \sigma(l_0)_t)$. We refer to the suspension $(\sum(V_0, E_0, l_0), \sigma(l_0)_t)$ as the principal suspension.

We need to introduce several auxiliary suspensions $(\sum(V_a, E_a, l_a), \sigma(l_a)_t)$, $a=1,\dots, \mu$, to cancel out overcounting of closed orbits. They are related to the principal suspension in the following manner: There exist maps π_a: $V_a \to V_0$ (*caution*: they are not morphisms of graphs) and Π_a: $\Sigma(V_a, E_a) \to \Sigma(V_0, E_0)$ such that

(a) $\Pi_a(\zeta)^0 = \pi_a(\zeta^0)$,

(b) for some positive integer $k = k(\zeta)$.

$$l_0(\Pi_a(\zeta)) = \sum_{i=0}^{k-1} l_a(\sigma^i \zeta),$$

(c) if we define a continuous map τ_a: $\Sigma(V_a, E_a, l_a) \to X$ by $\tau_a(\zeta, t) = \tau(\Pi_a(\zeta), t)$, then $\tau_a \circ \sigma(l_a)_t = \phi_t \circ \tau_a$,

(d) the topological entropy of $(\Sigma(V_a, E_a, l_a), \sigma(l_a)_t)$ is strictly less than h.

A counting lemma due to Manning and Bowen says that

$$\# P(\sigma(l_0)_t, m, x) + \sum_{a=1}^{\mu} (-1)^{q(a)} \# P(\sigma(l_a)_t, m, x)$$
$$= \begin{cases} 1 & \text{if} \quad m = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where, for example, $P(\sigma(l_a)_t, m, x)$ is the set of all $\sigma(l_a)_t$-closed orbits containing some points of $\tau_a^{-1}(x)$ whose period is $m \times$ (the period of x), and $q(a)$ are appropriate integers.

For a closed path $c = (v_0, \ldots, v_m)$ in (V_a, E_a), we define $\xi(c) \in \Sigma(V_a, E_a)$ by $\xi(c)^i = v_k$, where $i \equiv k \bmod m$. Then $(\xi(c), 0) \in \Sigma(V_a, E_a, l_a)$ is a periodic point of $\sigma(l_a)_t$. We then set

$$\mathfrak{p}(c) = \{(\xi(c), t)\}_{0 \leqslant t \leqslant t(\mathfrak{p}(c))},$$
$$t(\mathfrak{p}(c)) = \sum_{j=0}^{m-1} l_a(\sigma^j \xi(c)),$$
$$\Phi_a(c) = \tau_a(\mathfrak{p}(c)).$$

Φ_a is a map of the set of closed paths in (V_a, E_a) into the set of closed curves (in fact multiples of closed orbits) in X. We require the following lemma.

LEMMA 6-2. *There exists an embedding ι_a: $|(V_a, E_a)| \to X$ such that for any closed path c in (V_a, E_a), $\iota_a(c)$ is free homotopic to $\Phi_a(c)$.*

Proof. Select a point $x(v) \in \pi_a(v)$ for each $v \in V_a$ (for case $a = 0$, we take $x(v)$ in V_0). We may suppose that the distance between $x(v)$ and $x(w)$ is short whenever $(v, w) \in E_a$ (if necessary, we choose a Markov family with sufficiently small size), so that one can join them by a unique minimal geodesic, say $\gamma_a(v, w)$. Define a continuous map ι_a: $(V_a, E_a) \to X$ by

$$\iota_a(v) = x(v), \qquad \iota_a(v, w) = \gamma_a(v, w).$$

In the case $a = 0$, it is easy to see that $\iota_0(c)$ and $\Phi_0(c)$ are free homotopic for every closed path $c = (v_0, ..., v_m)$ in (V_0, E_0) since $x(v_j)$ and $\tau(\sigma^j\xi(c), 0)$ are contained in the same v_j. We now let $a \geqslant 1$, and let $c = (v_0, ..., v_m)$ be a closed path in (V_a, E_a). We denote by $c_a = (w_0, ..., w_n)$ a closed path in (V_0, E_0) which is uniquely determined by the relations $\Pi_a(\xi(c)) = \xi(c_a)$ and $\Phi_0(c_a) = \Phi_a(c)$. In view of the properties (b) and (c), $\gamma_0(w_j, w_{j+1})$ and the geodesic chain $\gamma_a(v_{i_j}, v_{i_j+1}) \cdot \gamma_a(v_{i_j+1}, v_{i_j+2}) \cdots \gamma_a(v_{i_{j+1}-1}, v_{i_{j+1}})$ are contained in a contractible ball, where i_j is inductively defined by

$$i_0 = 0, \qquad i_j = i_1 + \cdots + i_{j-1} + k(\sigma^{i_1 + \cdots + i_{j-1}}\xi(c)),$$

hence $\iota_a(c)$ and $\iota_0(c_a)$ are free homotopic. Since $\Phi_a(c) = \Phi_0(c_a)$, we conclude that $\iota_a(c)$ is free homotopic to $\Phi_a(c)$, as required.

We define a homomorphism φ_a to be the composition

$$\varphi_a: \pi_1(\Sigma^+(V_a, E_a)) \longrightarrow \pi_1(V_a, E_a) \xrightarrow{(\iota_a)_*} \pi_1(X).$$

Note that there are the following natural correspondences:

$$\begin{aligned} &\{\text{closed paths } c \text{ in } \Sigma^+(V_a, E_a) \text{ with } |c| = m\} \\ &\simeq \{\text{closed paths } c \text{ in } (V_a, E_a) \text{ with } |c| = m\} \\ &\simeq \{\xi \in \Sigma(V_a, E_a);\ \sigma^m\xi = \xi\} \\ &\simeq \{(\xi, 0) \in \Sigma(V_a, E_a, l_a);\ \sigma(l_a)_{t(\xi)}(\xi, 0) = (\xi, 0)\}, \end{aligned}$$

where $t(\xi) = \sum_{j=0}^{m-1} l_a(\sigma^j\xi)$. If $\mathfrak{p}$ is a $\sigma(l_a)_t$-closed orbit such that $\tau_a(\mathfrak{p})$ is an m-multiple of a ϕ_t-closed orbit $\mathfrak{q}$ then $\varphi_a(\langle \mathfrak{p} \rangle) = \langle \mathfrak{q}^m \rangle$ as conjugacy classes in $\pi_1(X)$ in view of Lemma 6-2. Therefore

$$\sum_{\mathfrak{p} \in P(\sigma(l_0)_t, m, x)} \operatorname{tr} \rho \circ \varphi_0(\langle \mathfrak{p} \rangle) + \sum_{\mathfrak{p} \in P(\sigma(l_a)_t, m, x)} (-1)^{q(a)} \operatorname{tr} \rho \circ \varphi_a(\langle \mathfrak{p} \rangle) = \begin{cases} \operatorname{tr} \rho(\langle \mathfrak{q} \rangle) & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if one sets

$$(\mathbf{V}_a, \mathbf{E}_a) = \Sigma(V_a, E_a),$$
$$l_a^+ = \text{a function on } \Sigma^+(V_a, E_a) \text{ cohomologous to } l_a,$$

this leads us to the conclusion of Proposition 6-1.

Theorem A is a consequence of this proposition and Theorem D. The assertion of Theorem B comes from the observation that if a representation $\rho\colon G \to U(N)$ is irreducible, then the composition $\pi_1(\Sigma^+(V_0, E_0)) \xrightarrow{\varphi_0} \pi_1(X) \xrightarrow{\varphi} G \xrightarrow{\rho} U(N)$ is also irreducible, since $\varphi \circ \varphi_0$ is surjective from the assumption.

Remark 1. Suppose that the image of a representation $\rho\colon \pi_1(X) \to U(N)$, say G, is of finite order. Let $(\hat{X}, \phi_t) \to (X, \phi_t)$ be a finite cover corresponding to $\operatorname{Ker} \rho$. The lifting $(\hat{V}_0, \hat{E}_0)$ of a Markov family (V_0, E_0) to $\hat{X}$ is a Markov family of $(\hat{X}, \phi_t)$, so that we have a normal covering map: $|(\hat{V}_0, \hat{E}_0)| \to |(V_0, E_0)|$ of connected CW-complexes whose covering transformation group is G. Thus we have a surjective homomorphism $\pi_1(V_0, E_0) \to G$ which is factorized as

$$\begin{tikzcd} \pi_1(V_0, E_0) \arrow[rr] \arrow[dr] & & G \\ & \pi_1(X) \arrow[ur] & \end{tikzcd}$$

Remark 2. As a generalization of Selberg zeta functions, R. Gangolli [7] introduced a zeta function associated to a geodesic flow on a compact locally symmetric space with negative curvature, which has the form

$$Z(s, \rho) = \prod_{\mathfrak{p}} \prod_{k_1, \dots, k_{n-1} = 0}^{\infty} \det(I - \rho(\langle \mathfrak{p} \rangle) \times \exp(-(s + k_1 r_1 + \cdots k_{n-1} r_{n-1})\, l(\mathfrak{p}))),$$

where n denotes the dimension of the symmetric space and r_i^2 are the positive egenvalues of the curvature operator: $u \to R(u, v)\, v$, v being a unit tangent vector. He in fact showed that all the results established by Selberg for the surface case are generalized to higher dimensional spaces. A relation between the Gangolli's zeta function and our L-function is

$$L(s, \rho) = \frac{\prod_i Z(s+i, \rho) \prod_{i_1 < i_2 < i_3} Z(s + r_{i_1} + r_{i_2} + r_{i_3}, \rho) \cdots}{Z(s, \rho) \prod_{i_1 < i_2} Z(s + r_{i_1} + r_{i_2}, \rho) \cdots}$$

Acknowledgments

We acknowledge the many fruitful discussions which we had with A. Katsuda while working on this paper.

Note added in proof. The first author [28] recently showed that closed orbits of a transitive Anosov flow generate the fundamental group of the base manifold. Hence in Theorem B we may only assume $\varphi\colon \pi_1(X) \to G$ is surjective.

References

1. T. Adachi and T. Sunada, Homology of closed geodesics in a negatively curved manifold, *J. Differential Geom.*, to appear.
2. R. Bowen, Symbolic dynamics for hyperbolic flows, *Amer. J. Math.* **95** (1973), 429–460.
3. R. Bowen, "Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms," Lecture Note in Mathematics, Vol. 470, Springer-Verlag, New York, 1975.
4. D. Fried, The zeta functions of Ruelle and Selberg, I, preprint, 1985.
5. D. Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, preprint, 1985.
6. D. Fried, Fuchsian groups and Reidemeister torsion, preprint, 1985.
7. R. Cangolli, Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one, *Illinois J. Math.* **21** (1977), 1–42.
8. D. A. Hejhal, The Selberg trace formula and the Riemann zeta function, *Duke Math. J.* **43** (1976), 441–482.
9. D. A. Hejhal, "The Selberg Trace Formula for $PSL_2(\mathbb{R})$," Springer Lecture Notes 548, Vol. ; 1001, Vol. II, Springer-Verlag, New York.
10. Y. Ihara, Discrete subgroups of $PL(2, k_p)$, *Proc. Sympos. Pure Math.* **9** (1966), 272–278.
11. A. Manning, Axiom A diffeomorphisms have rational zeta functions, *Bull. London Math. Soc.* **3**) (1971), 215–220.
12. H. P. McKean, Selberg's trace formula as applied to a compact Riemann surface, *Comm. Pure Appl. Math.* **25** (1972), 225–246.
13. W. Farry and M. Pollicott, An analogue of the prime number theorem for closed orbits of Axiom A flows, *Ann. of Math.* **118** (1983), 573–591.
14. W. Parry and M. Pollicott, The Chebotarev theorem for Galois coverings of Axiom A flows preprint.
15. J. F. Plante, Homology of closed orbits of Anosov flows, *Proc. Amer. Math. Soc.* **37** (197[illegible]), 297–300.
16. M. Pollicott, A complex Ruelle–Perron–Frobenius theorem and two counter-examples, preprint.
17. M. Pollicott, Meromorphic extensions of generalized zeta functions, preprint.
18. D. Ruelle, "Thermodynamic Formalism," Addison–Wesley, Reading, Mass., 1978.
19. D. Ruelle, Zeta functions for expanding maps and Anosov flows, *Invent. Math.* **34** (1976), 231–242.
20. P. Sarnak, The arithmetic and geometry of some hyperbolic three manifolds, *Acta Math.* **151** (1983), 253–296.
21. A. Selberg, Harmonic analysis and discontinuous subgroups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc.* **20** (1956), 47–8[illegible].
22. J. P. Serre, "Trees," Springer-Verlag, New York, 1980.
23. S. Smale, Differential dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747–817.
24. T. Sunada, Geodesic flows and geodesic random walks, *in* "Geometry of Geodesics and Related Topics," Advanced Studies in Pure Mathematics, Vol. 3, 1984.
25. T. Sunada, Tchebotarev's density theorem for closed geodesics in a compact locally symmetric space of negative curvature, preprint.
26. T. Sunada, Riemannian coverings and isospectral manifolds, *Ann. of. Math.* **121** (1985), 169– 86.
27. T. Sunada, Trace formulas, Wiener integrals and asymptotics, *in* "Proceedings, Japan–France Seminar, Spectra of Riemannian Manifolds," pp.103–113, Kaigai, Tokyo, 1983.
28. T. Adachi, Closed orbits of an Anosov flow and the fundamental group, *Proc. Amer. Math. Soc.*, to appear.